Limit shapes of pattern-avoiding Mallows permutations

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M. Slim Kammoun, and Mylène Maïda

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1 Pattern avoidance

2 Mallows models

3 Limit shapes

4 Strategy

5 The 231 case

6 The 321 case

Pattern-avoiding permutations

Patterns

If $\sigma = \sigma_1 \dots \sigma_n$ is a permutation and $I = \{i_1 < \dots < i_k\}$ are indices, we let $\mathrm{pat}(I, \sigma)$ denote the (unique) permutation $\alpha = \alpha_1 \dots \alpha_k$ with the same relative ordering as $\sigma_{i_1}, \dots, \sigma_{i_k}$:

$$\forall a, b, \quad \alpha_a < \alpha_b \iff \sigma_{i_a} < \sigma_{i_b}\,.$$

$$\left.\begin{array}{l} \sigma = \ 3\ 6\ 1\ 5\ 2\ 4 \\ I = \{1, 4, 5\} \end{array}\right\} \mathrm{pat}(I, \sigma) =$$

Avoidance class

For a given $\alpha \in \mathfrak{S}$, we let $\mathrm{Av}_n(\alpha)$ denote the set of α-avoiding n-sized permutations.

$\longrightarrow$ Size of $\mathrm{Av}_n(\alpha)$? Structure?
What does a typical element look like?

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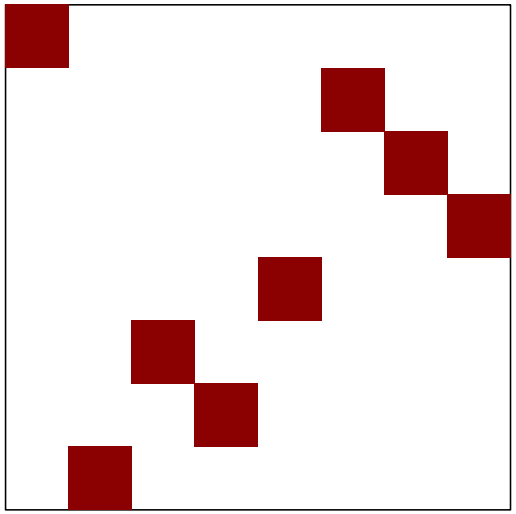
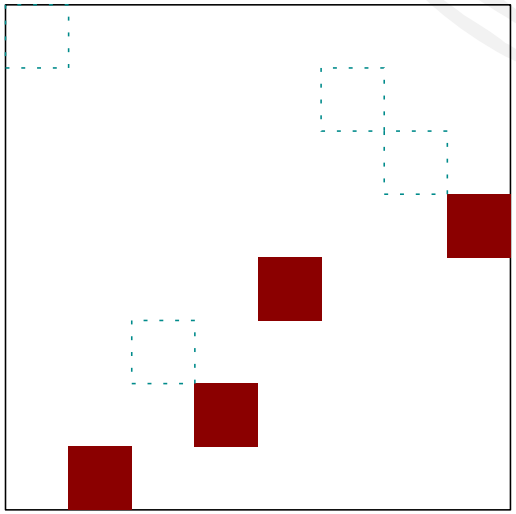
The sequence of RLMs

Right-to-left-minima

A right-to-left minimum (RLM) of $\sigma \in \mathfrak{S}_n$ is a point $(i, \sigma(i))$ such that $\sigma(i) > \sigma(j)$ for all $j > i$.



$\sigma = 81324765$



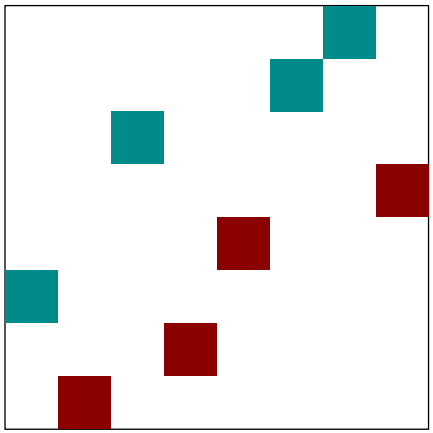
321-avoiding permutations

Structure of the 321 avoidance class

- $\sigma \in \mathrm{Av}_n(321)$ is a union of two increasing subsequences. Entirely described by a specific one, its list of RLMs.
- $\mathrm{Av}_n(321) \longleftrightarrow \mathrm{Dyck}_n := \{\text{Dyck paths of length } 2n\}$.

$$F_\sigma : x \in [0, n) \mapsto \inf\{\sigma(i) : i > x\} - 1$$

Size: $C_n = \frac{1}{n+1}\binom{2n}{n}$.



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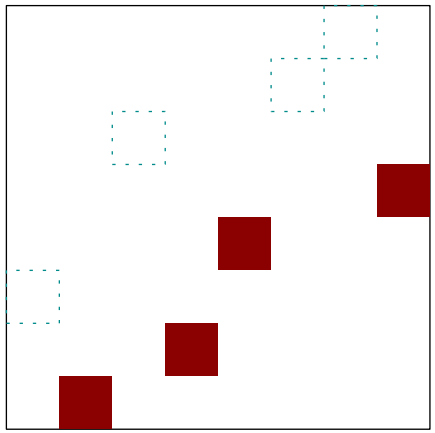

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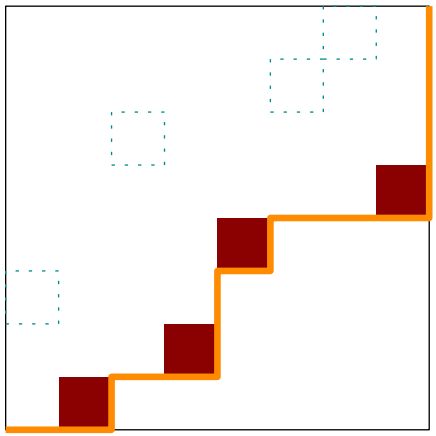
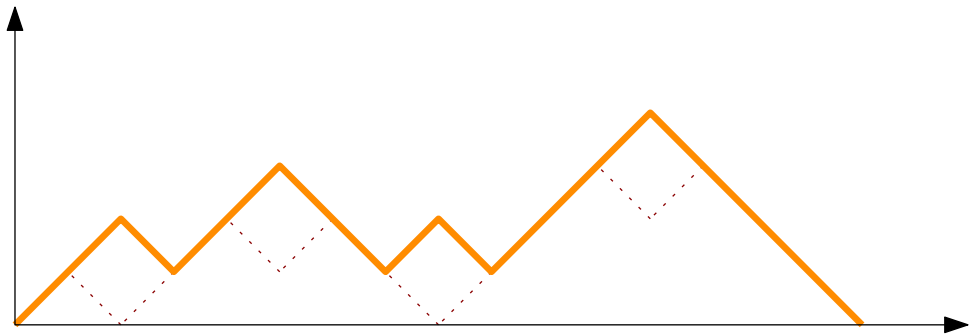




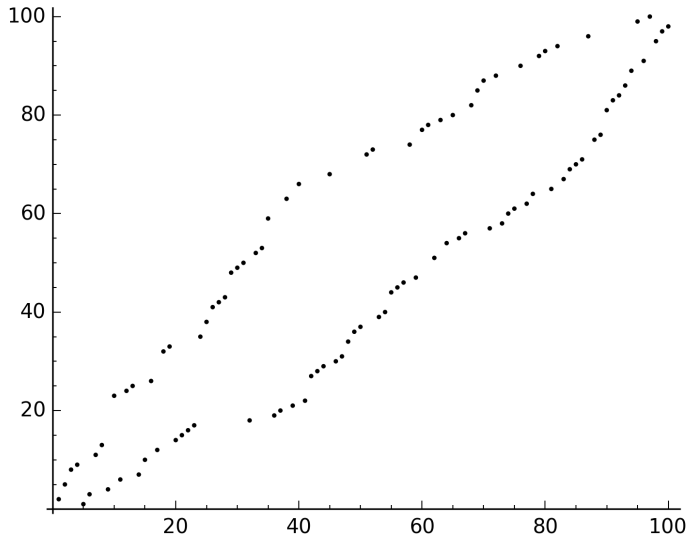
321-avoiding permutations

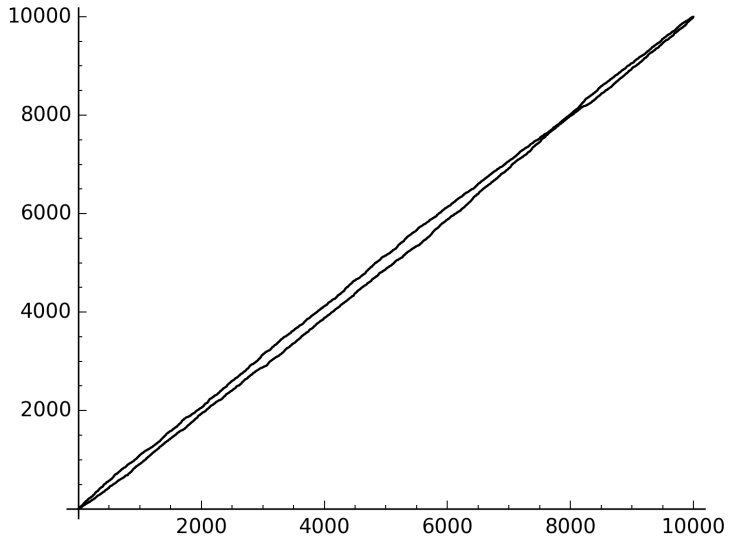
Typical behavior of 321-avoiding permutations

If $\sigma_n \sim \mathrm{Unif}(\mathrm{Av}_n(321))$ then $\sigma_n(i) \approx i \pm \sqrt{2n}\,\mathfrak{e}(i/n)$

$\mathfrak{e}$: Brownian excursion.

[Miner–Pak'14], [Madras–Pehlivan'15], [Hoffman–Rizzolo–Slivken'17]





231-avoiding permutations

Structure of the 231 avoidance class

- $\mathrm{Av}_n(231)$ is the class of stack-sortable permutations.
- $\sigma \in \mathrm{Av}_n(231)$ is entirely described by its RLMs.
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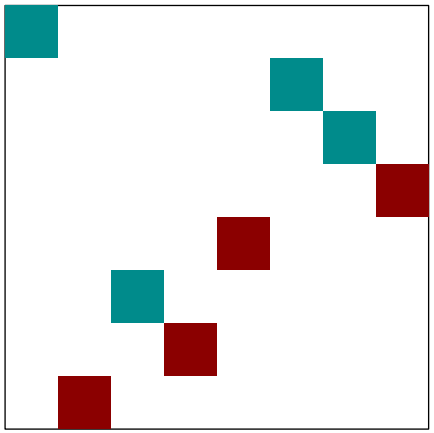

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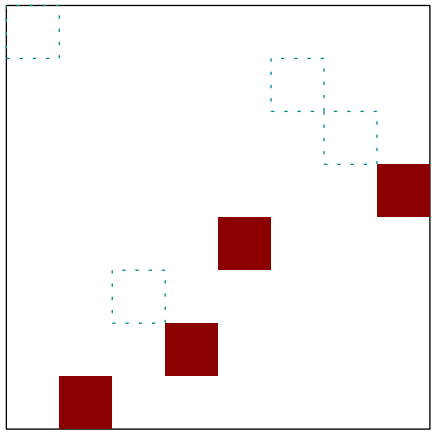

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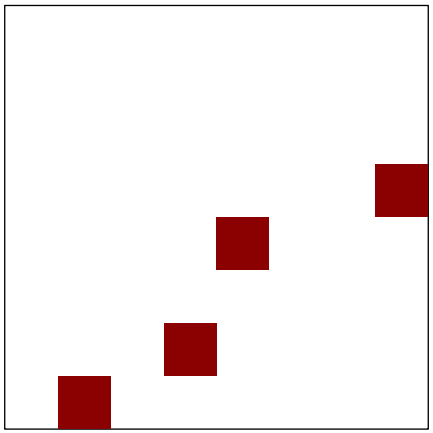

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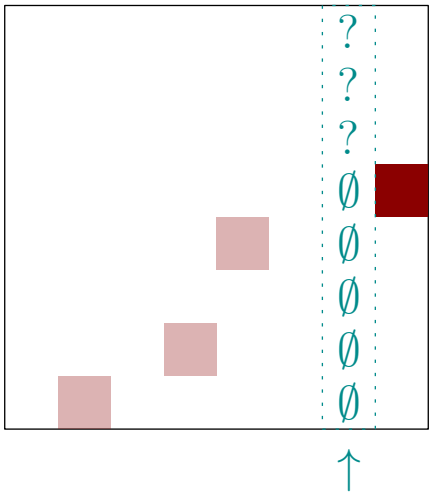

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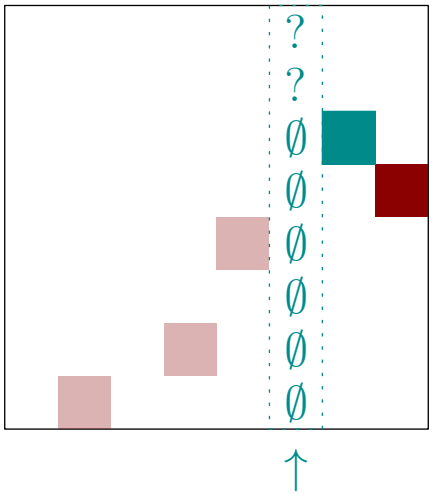

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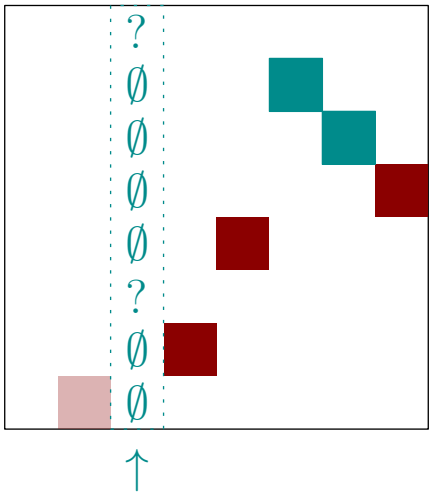

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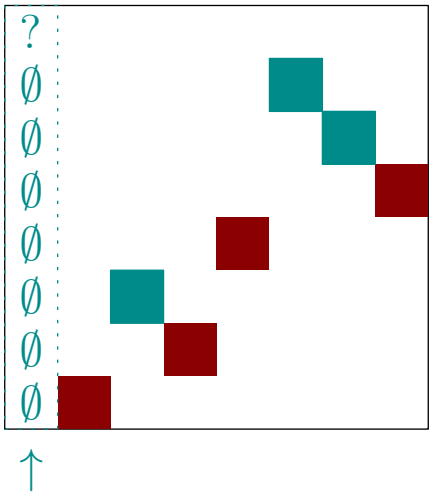

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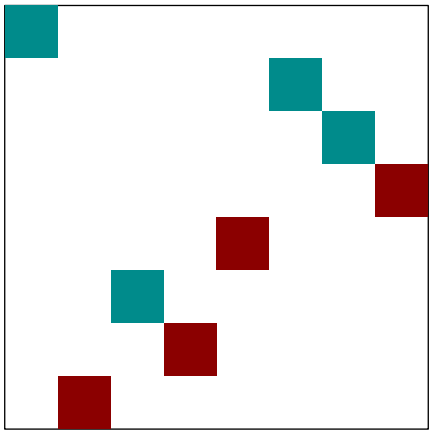

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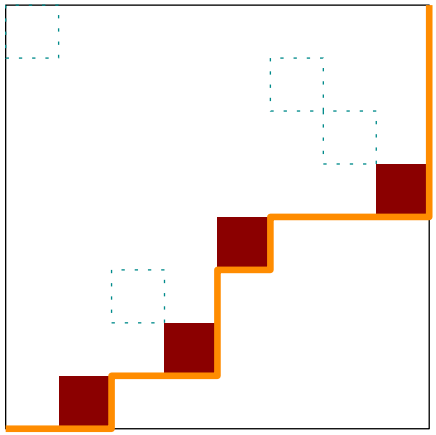

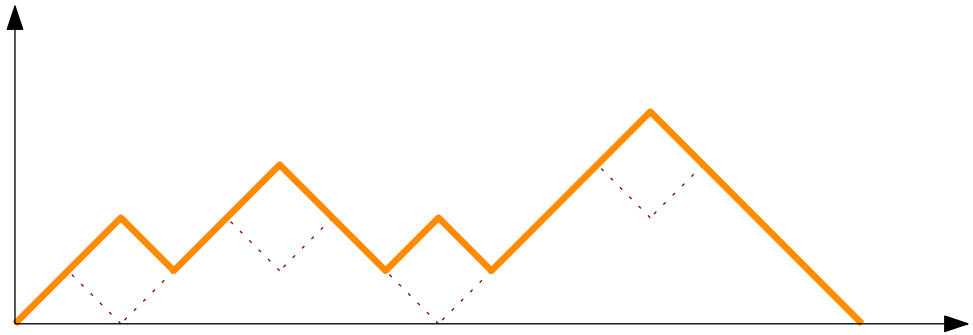

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Typical behavior of a 231-avoiding permutation

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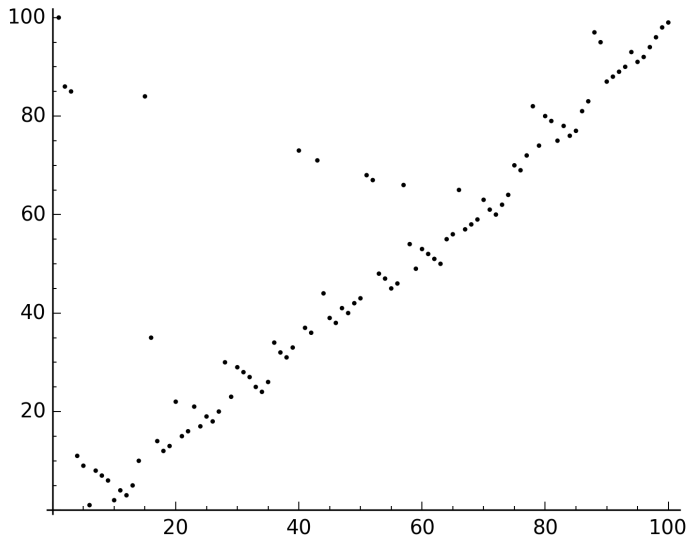

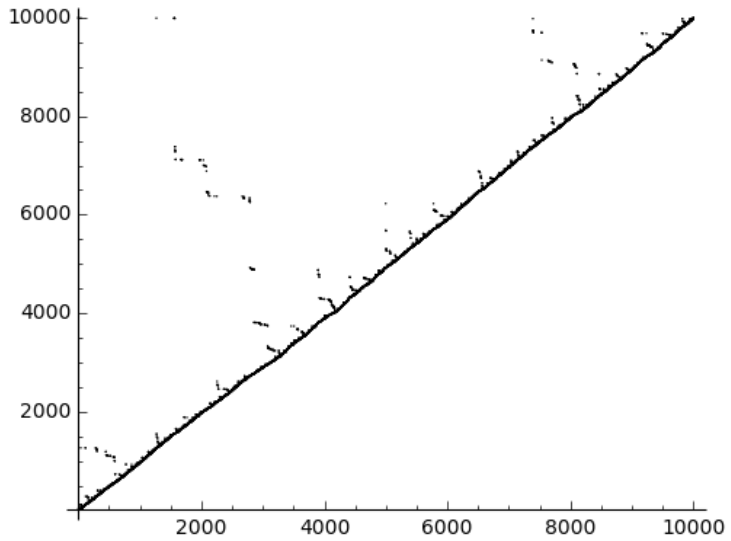

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Mallows random permutations

inversion: occurrence of a 21 pattern

$\mathrm{inv}(\sigma)$ = number of inversions in σ

The unconstrained Mallows distribution

Let $q > 0$. $\tau_n \sim \mathrm{Mallows}(n, q)$ if:

$$\forall \sigma \in \mathfrak{S}_n, \quad \mathbb{P}(\tau_n = \sigma) = \frac{1}{Z_{n,q}} q^{\mathrm{inv}(\sigma)}$$

where $Z_{n,q} = \prod_{k=1}^{n} \frac{1-q^k}{1-q}$.

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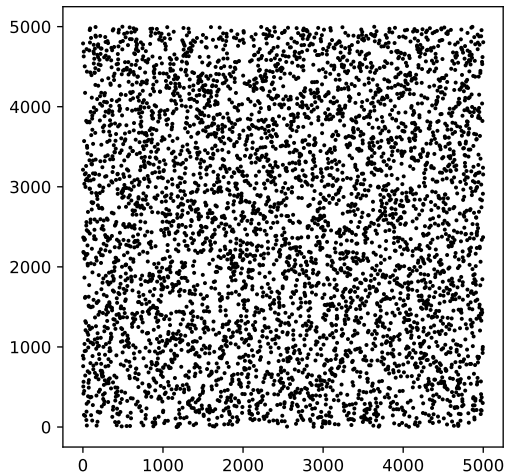

$q = 1$

Uniform

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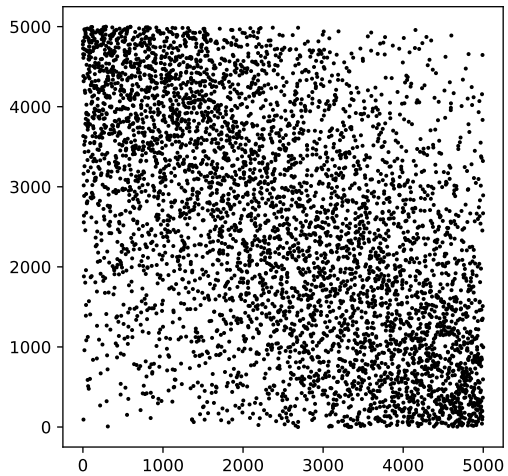

$q_n = 1 + 4/n$

Phase transition

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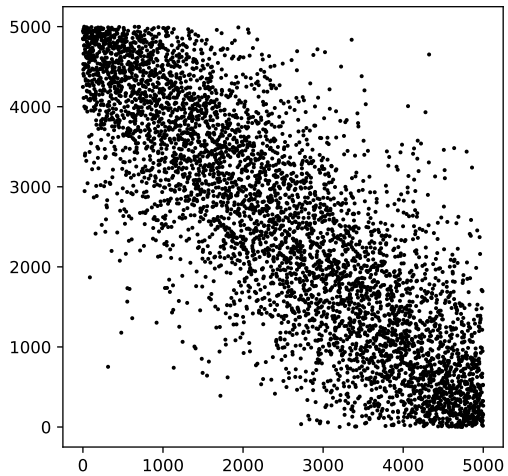

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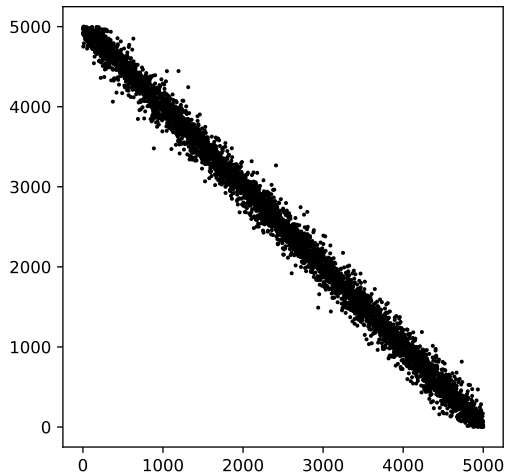

$$q_n = 1 + 1/\sqrt{n}$$

Almost antidiagonal

Our model: pattern-avoiding Mallows permutations

The constrained Mallows distribution

Let $\alpha \in \mathfrak{S}$ and $q > 0$. $\tau_n^{q,\alpha} \sim \mathrm{Mallows}(n, q, \alpha)$ if:

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where $Z_{n,q,\alpha} = \sum_{\sigma \in \mathrm{Av}_n(\alpha)} q^{\mathrm{inv}(\sigma)}$.

Focus on: $\alpha \in \{231, 321\}$* and $q = q_n = \exp(\beta/n)$

*encompasses all $\alpha \in \mathfrak{S}_3$

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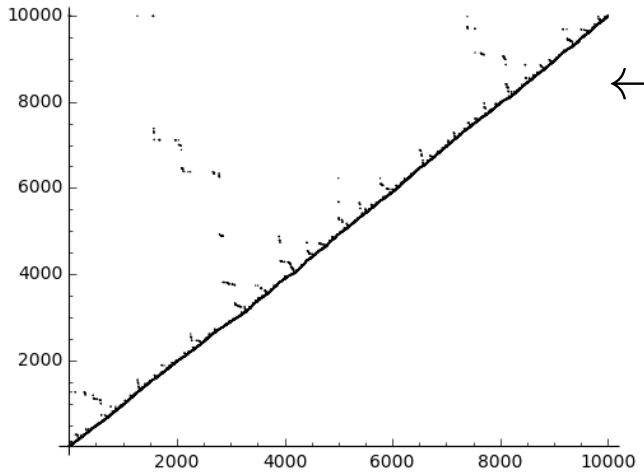
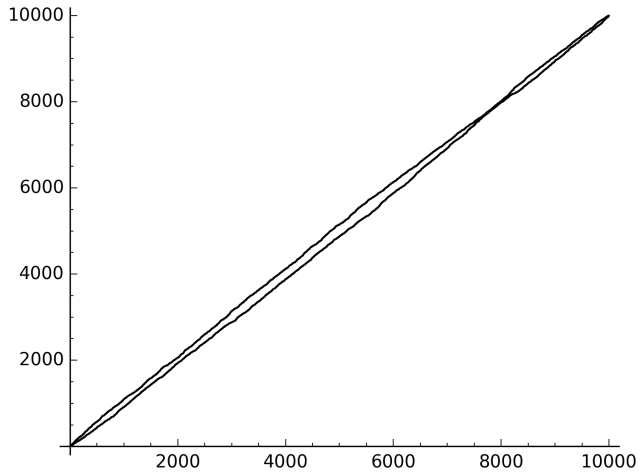


$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

$\beta = 0$



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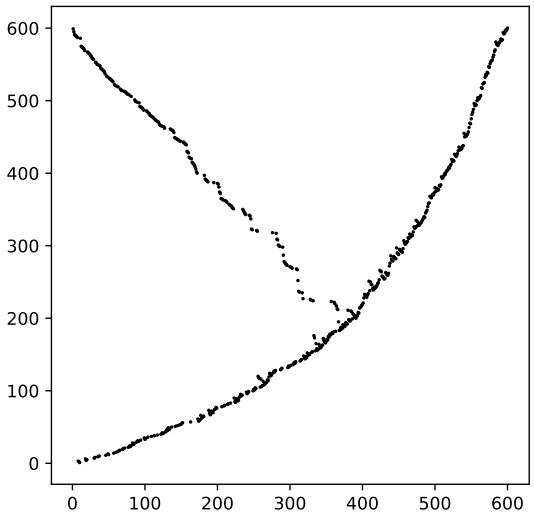
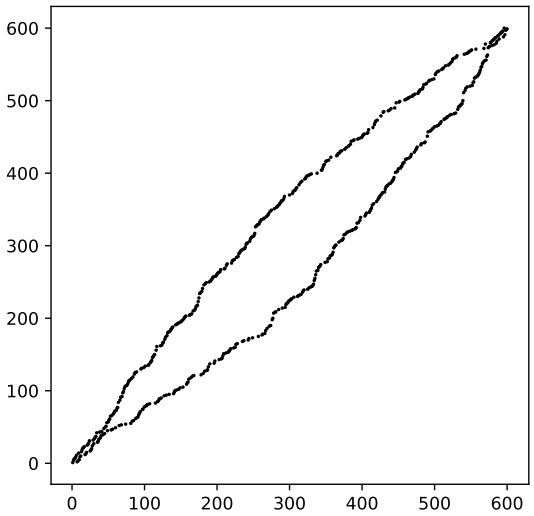
* encompasses all $\alpha \in \mathfrak{S}_3$



$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

$\beta = 1$



Our model: pattern-avoiding Mallows permutations

The constrained Mallows distribution

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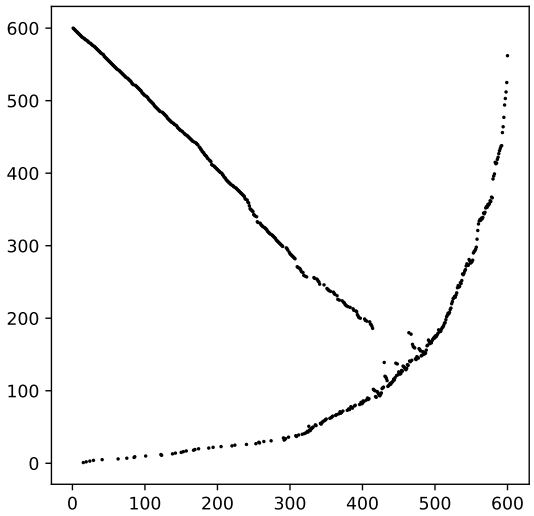
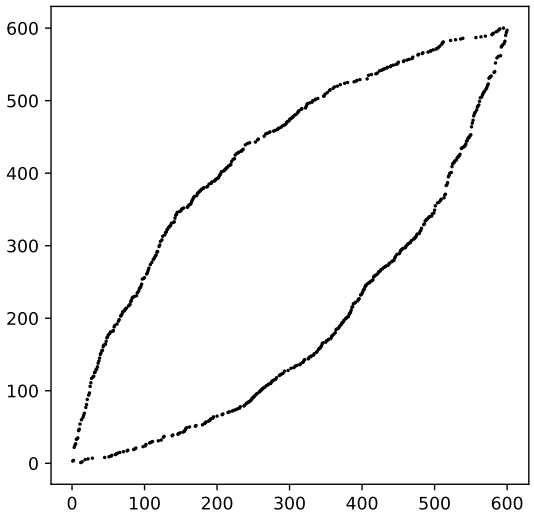


$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

$\beta = 3$



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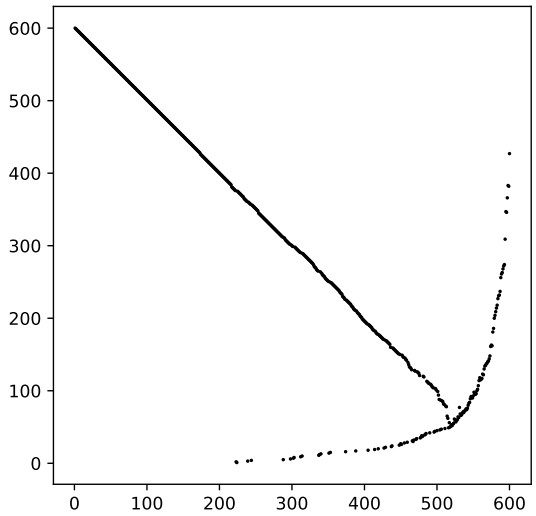
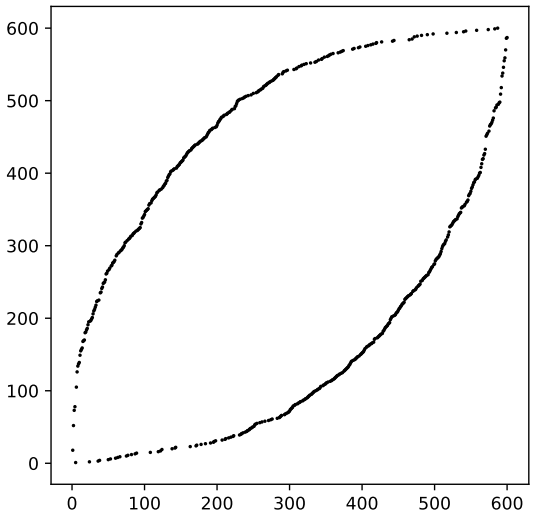


$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

$\beta = 6$



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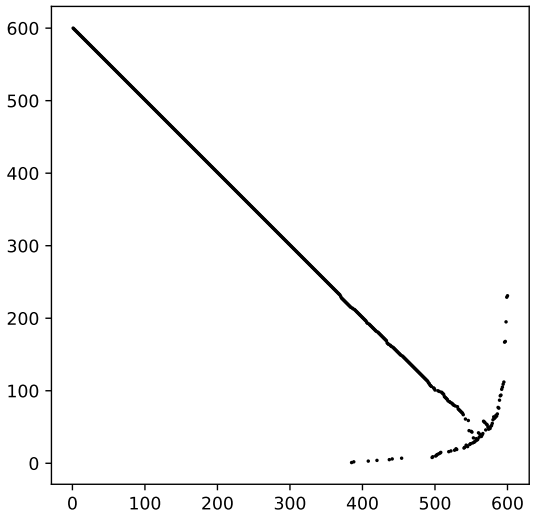
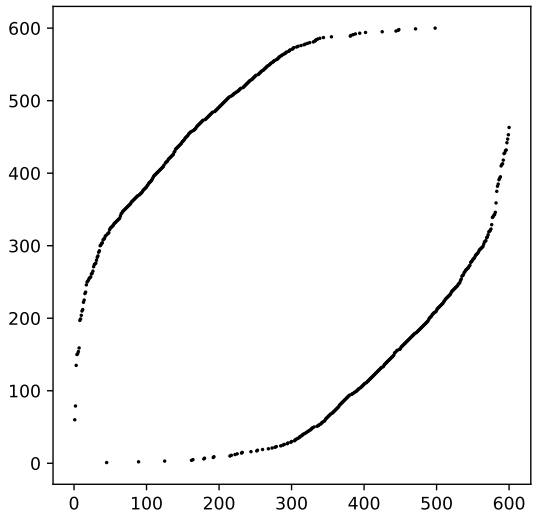


$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

$\beta = 12$



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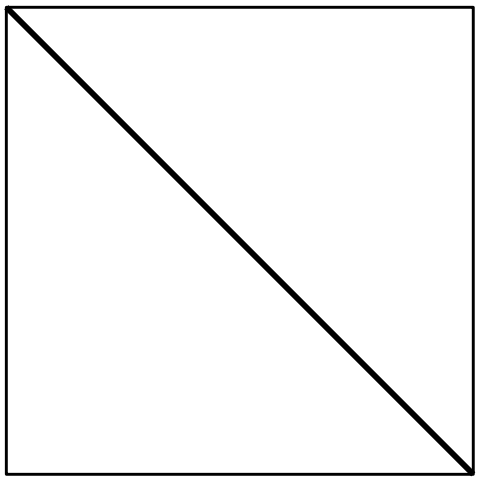
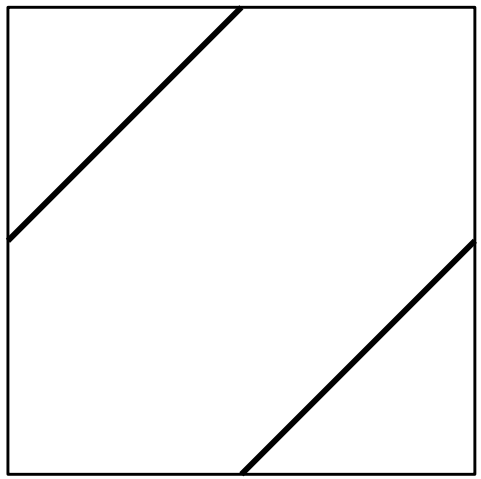


$\leftarrow \alpha = 231$

$\alpha = 321 \rightarrow$

"$\beta = \infty$"



1 Pattern avoidance

2 Mallows models

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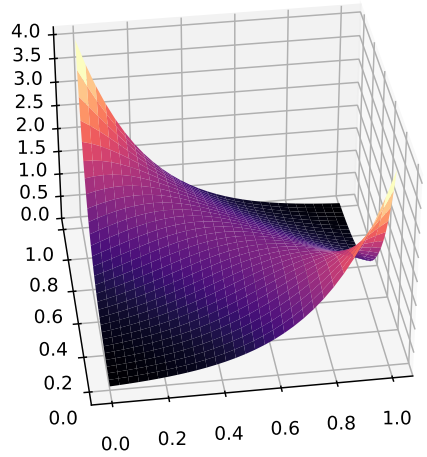
5 The 231 case

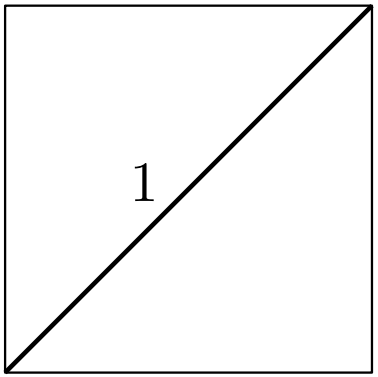
6 The 321 case

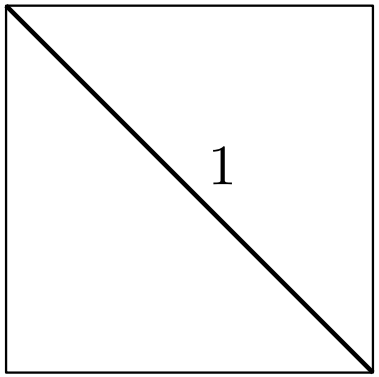
Permuton theory

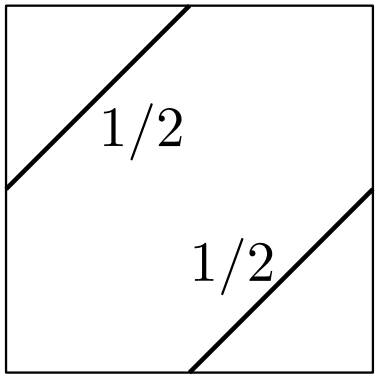
The space of permutons

A permuton is a probability measure $\mu \in \mathcal{M}_1([0,1]^2)$ with uniform marginals: $\forall t \in [0,1],\ \mu([0,t] \times [0,1]) = \mu([0,1] \times [0,t]) = t$.









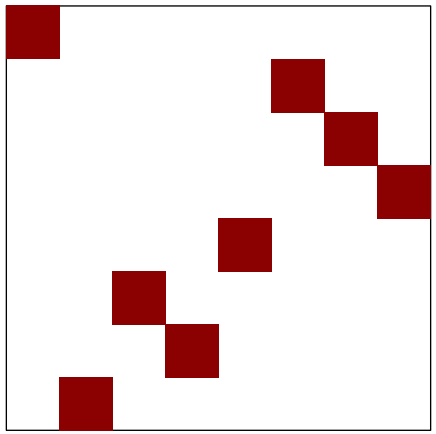
Permuton theory

Permutations as permutons

Each permutation $\sigma \in \mathfrak{S}_n$ is mapped to a permuton μ_σ with density

$$\frac{\mathrm{d}\mu_\sigma}{\mathrm{dLeb}} = n \sum_{i=1}^{n} \mathbb{1}_{[\frac{i-1}{n},\frac{i}{n}]\times[\frac{\sigma(i)-1}{n},\frac{\sigma(i)}{n}]}.$$

We say that $\sigma_n \longrightarrow \mu$ if $\mu_{\sigma_n} \longrightarrow \mu$ weakly.



$\sigma = 81324765$

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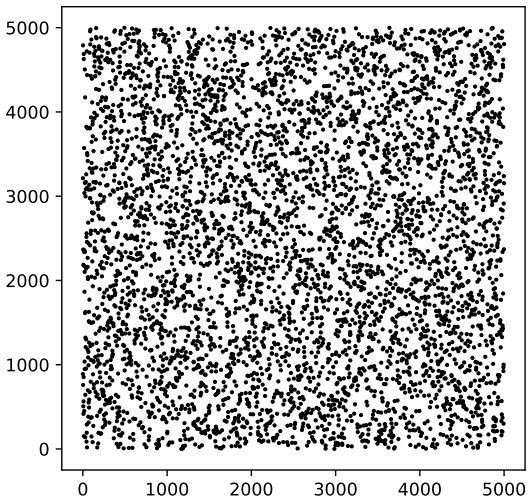
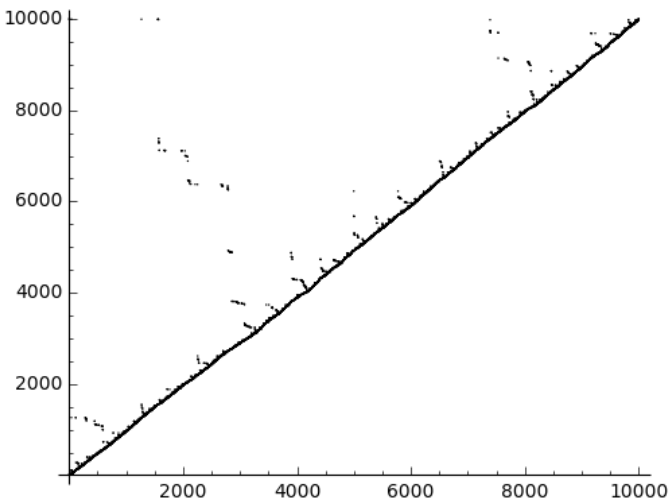


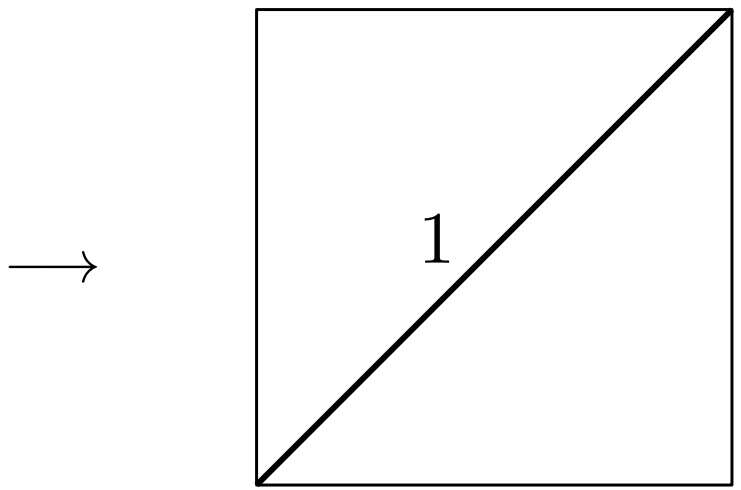
$\longrightarrow$

Permuton theory

Examples

- For $\alpha \in \{231, 321\}$, if $\sigma_n \sim \mathrm{Unif}(\mathrm{Av}_n(\alpha))$ then a.s. $\sigma_n \longrightarrow \mu_{/}$ (diagonal permuton).
- For $q = q_n = \exp(\beta/n)$, if $\tau_n \sim \mathrm{Mallows}(n, q)$ then a.s. $\tau_n \longrightarrow \mu_\beta$ with an explicit density [Starr'09].





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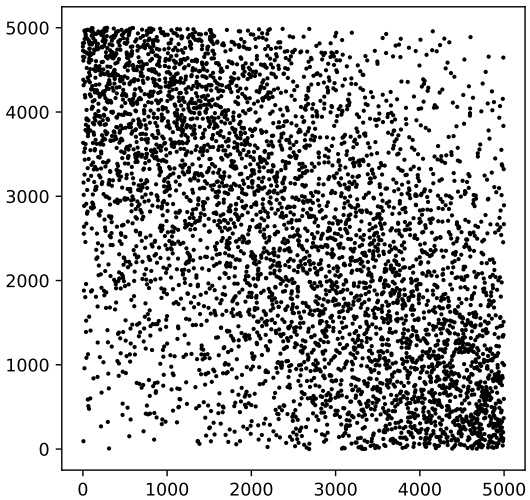
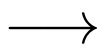
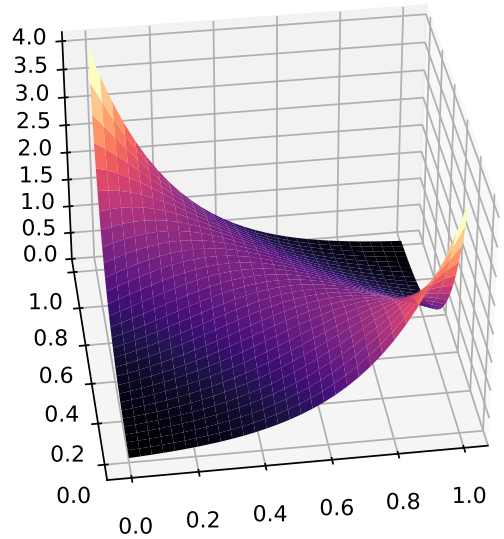






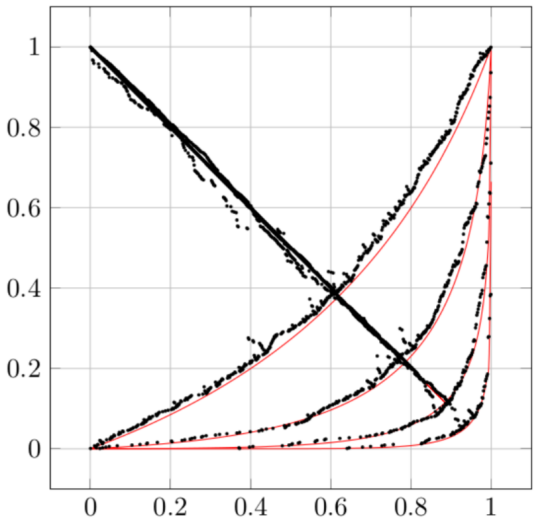
Main results

Limit shapes of 231-avoiding Mallows [BDFKM'26]

Let $\beta > 0$ and $q = \exp(\beta/n)$. Let

$$f_\beta^{231}(x) = 1 - \frac{1}{\beta}\log(1 + e^\beta - e^{\beta x})\,.$$

Then $\tau_n^{q,231} \xrightarrow{a.s.} \mu_\beta^{231}$ with density $1 \wedge (f_\beta^{231})'(x)\mathrm{d}x$ on the graph of f_β^{231} and complementary density on the antidiagonal.



$\beta \in \{1, 3, 6, 12\}$

Main results

Limit shapes of 321-avoiding Mallows [BDFKM'26]

Let $\beta > 0$ and $q = \exp(\beta/n)$. Let

$$f_\beta^{321}(x) = \frac{1}{2} + \frac{1}{\beta} \log\left(\frac{e^{\beta x} + e^{\beta/2}}{1 + e^{\beta/2} + e^{\beta} - e^{\beta x}} \right).$$

Then $\tau_n^{q,321} \xrightarrow{a.s.} \mu_\beta^{321}$ with density $\frac{\mathrm{d}x}{1+e^{\beta(\frac{1}{2}-x)}}$ on the graph of f_β^{321} and complementary density on its symmetric.

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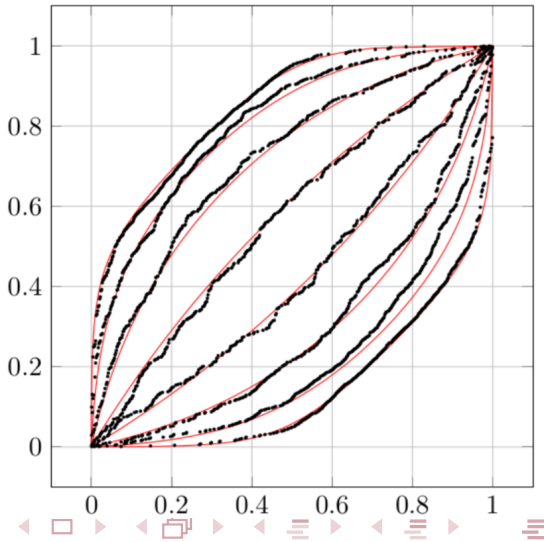

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The LDP approach

Context: Discrete space $\mathcal{E}_n \subset \mathcal{E}$ compact space.
Parameter $q_n > 0$, continuous statistic $h : \mathcal{E} \to \mathbb{R}_+$.

Goal: Asymptotics of the (q, h)-tilted measure on $\mathcal{E}_n$

$$\mu_{n,q}(x_n) = \frac{1}{Z_{n,q}} q^{h(x_n)}$$

Ingredient:
The uniform measure $\mu_{n,1}$ satisfies a *Large Deviation Principle* with speed a_n and rate function I: $\mu_{n,1}(x_n \approx x) \approx \exp[-a_n I(x)]$

Consequences:

- (Varadhan's lemma) For $q_n = e^{a_n}$, the tilted measure satisfies a LDP with speed a_n and rate function $I - h - \min(I - h)$:

$$\mu_{n,q}(x_n \approx x) \propto \exp[a_n h(x) - a_n I(x)]$$

- If rate function has a unique minimum, then a.s. convergence.

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Permuton avoidance classes

What are the right spaces for us?

Patterns and permutons

- For $\alpha \in \mathfrak{S}_3$, $\mathrm{dens}(\alpha, \mu)$ is the probability that 3 μ-i.i.d. points are in α position.
- Let $\mathrm{Av}_\square(\alpha)$ be the set of α-avoiding permutons, i.e., such that $\mathrm{dens}(\alpha, \mu) = 0$
- If $\sigma \in \mathrm{Av}_n(\alpha)$, μ_σ may be replaced with $\mu_\sigma^\alpha \in \mathrm{Av}_\square(\alpha)$.

$\longrightarrow$ In the study of α-avoiding permutations and their permuton limits, we can restrict our attention to $\mathrm{Av}_\square(\alpha)$.

$\longrightarrow$ We would like to find a "nice parametrization space"

$$\mathcal{E} \longleftrightarrow \mathrm{Av}_\square(\alpha)$$

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Parametrizations

Encoding of α-avoiding permutons [BDFKM'26]

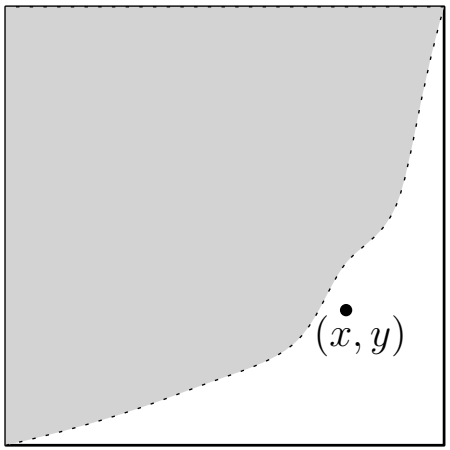
The RLM curve of $\mu \in \mathrm{Av}_\square(\alpha)$ is

$$f_\mu : x \in [0,1] \mapsto \max\{y \in [0,1] : \mu([x,1] \times [0,y]) = 0\}\,.$$

$\mu \in \mathrm{Av}_\square(\alpha) \mapsto (f_\mu, \mu_{\mathrm{graph}(f_\mu)})$ is one-to-one.

Problems:

- That's a bit "too much" information ("ugly" parametrization space)
- That's not continuous!



$$f(x) \geq y \quad \Leftrightarrow \quad \mu([x,1] \times [0,y]) = 0$$

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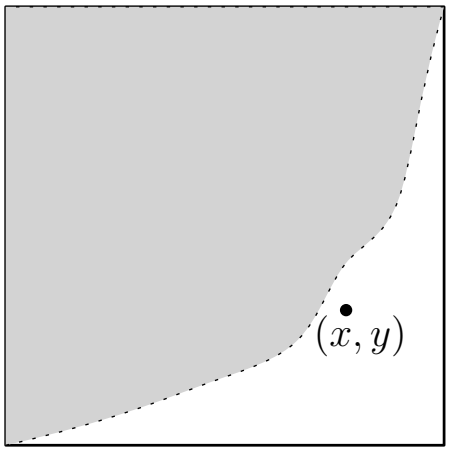

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Parametrizations

Encoding of α-avoiding permutons [BDFKM'26]

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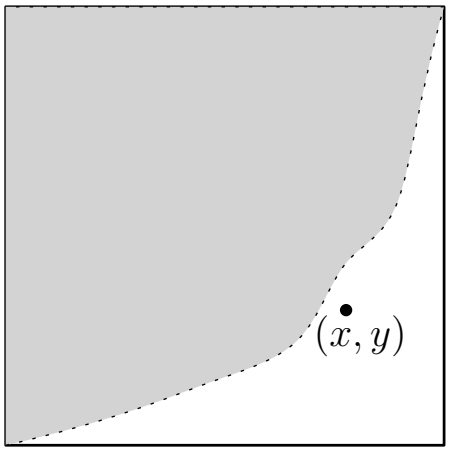

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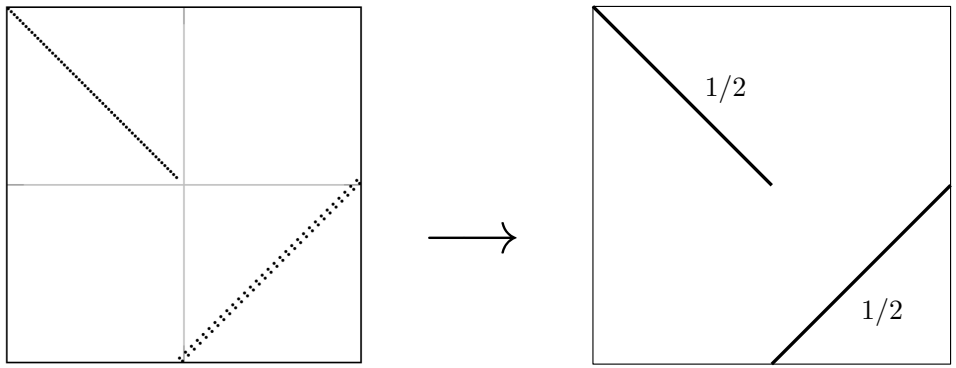
$\underline{\alpha = 231}$

Problem with the mass



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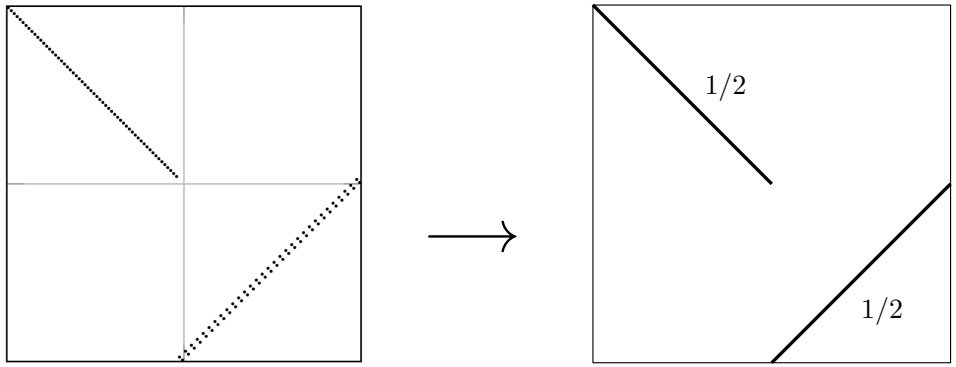

$\mu \in \mathrm{Av}_\square(231) \mapsto f_\mu$ is one-to-one and bicontinuous!

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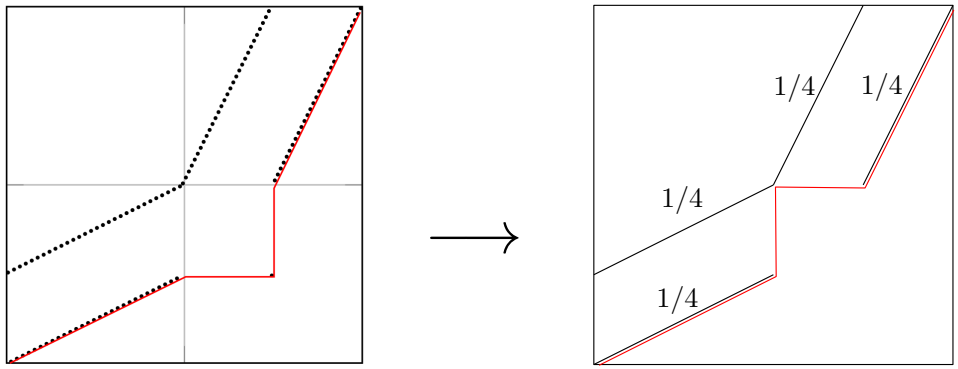

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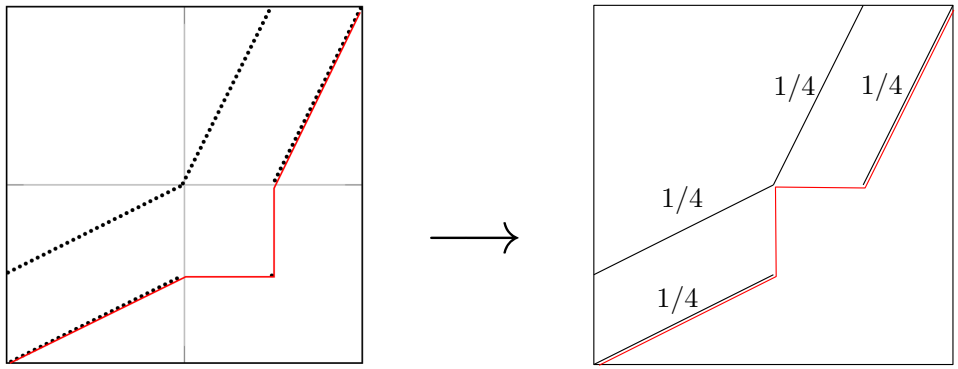


There is a continuous $\nu \twoheadrightarrow \mu \in \mathrm{Av}_\square(321)$, consistent with $\mu \mapsto \mu_{\mathrm{graph}(f_\mu)}$

1 Pattern avoidance

2 Mallows models

3 Limit shapes

4 Strategy

5 The 231 case

6 The 321 case

Parametrization of 231-avoiding permutations

$$\sigma \in \mathrm{Av}_n(231)$$

$\updownarrow$ *RLM list*

$$\{(i, \sigma(i)) : \forall j > i, \sigma(j) > \sigma(i)\}$$

$\updownarrow$ *RLM curve*

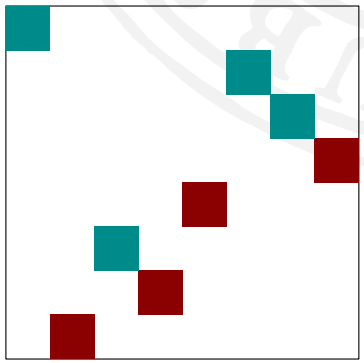
$$F_\sigma : x \in [0, n) \mapsto \min\{\sigma(i) : i > x\} - 1$$

$\updownarrow$ *rotation*

$$\Phi_\sigma \in \mathrm{Dyck}_n$$

$\updownarrow$ *rescaling*

$$\varphi_\sigma \in \{1\text{-Lipschitz excursions on } [0,1]\} = \mathcal{E}$$



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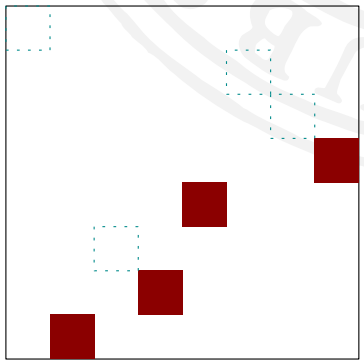

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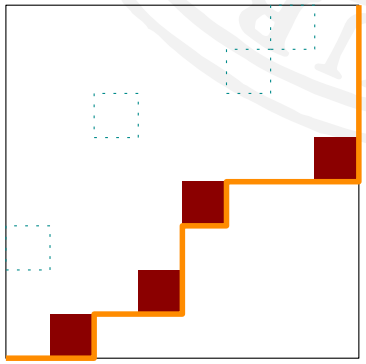

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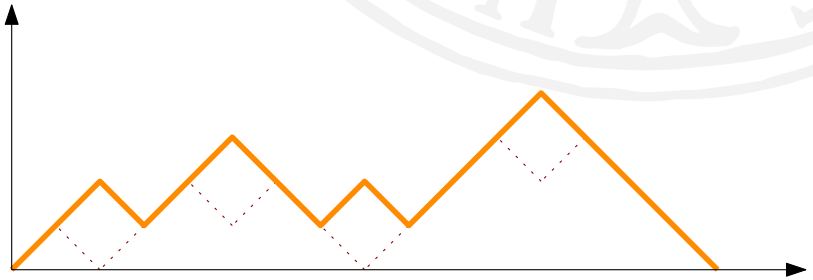

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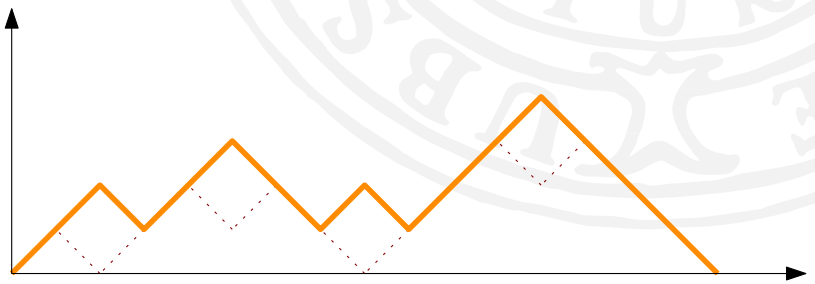

Inversion number

$$\mathrm{inv}(\sigma) = \tfrac{-1}{2} n + 2n^2 \int \varphi_\sigma$$

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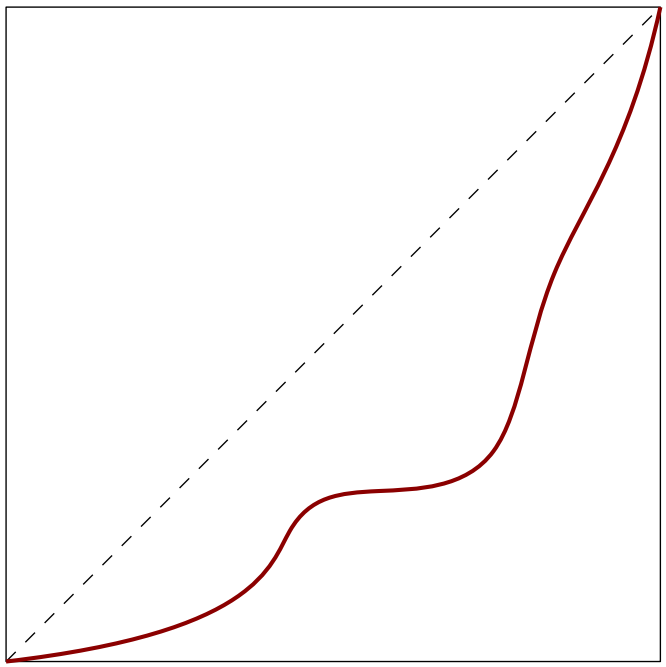

μ?

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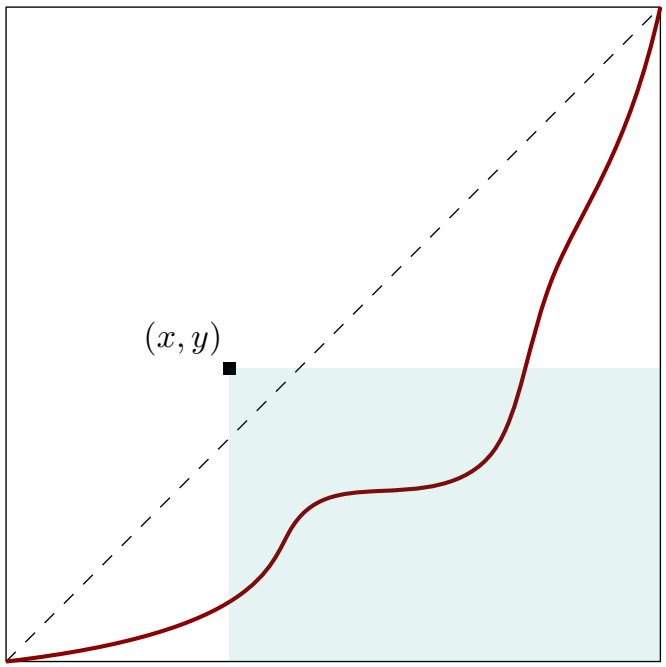

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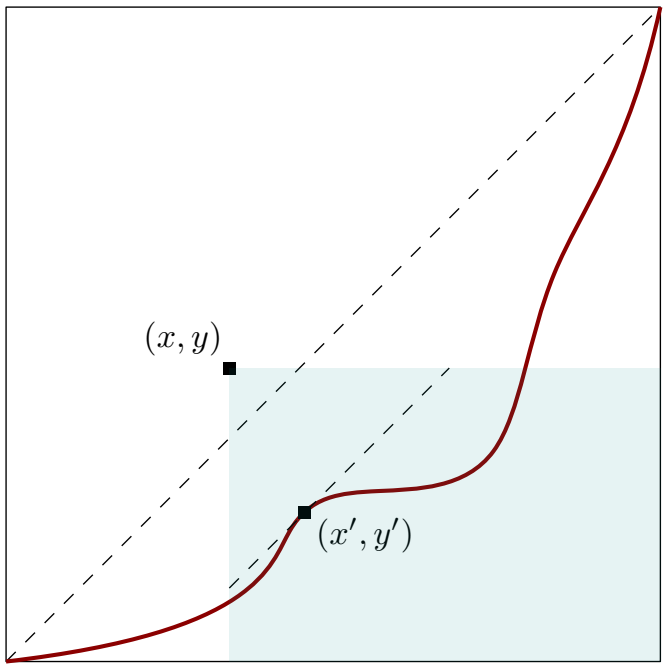

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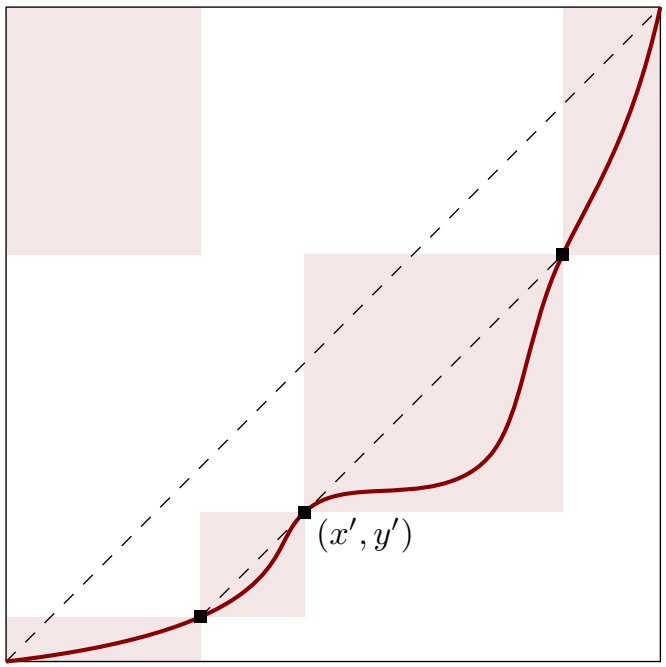

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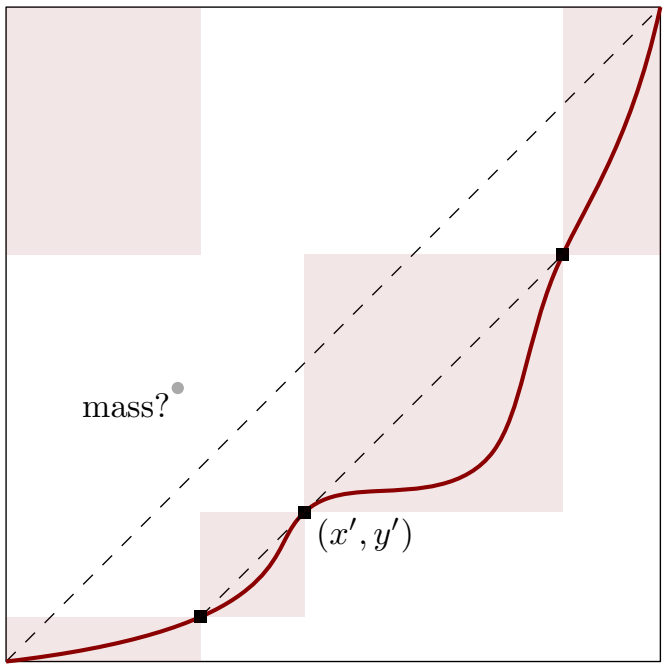

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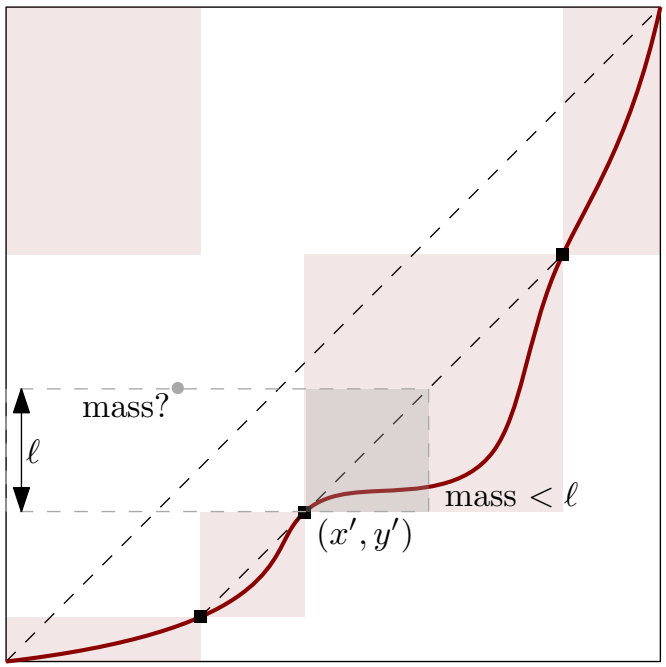

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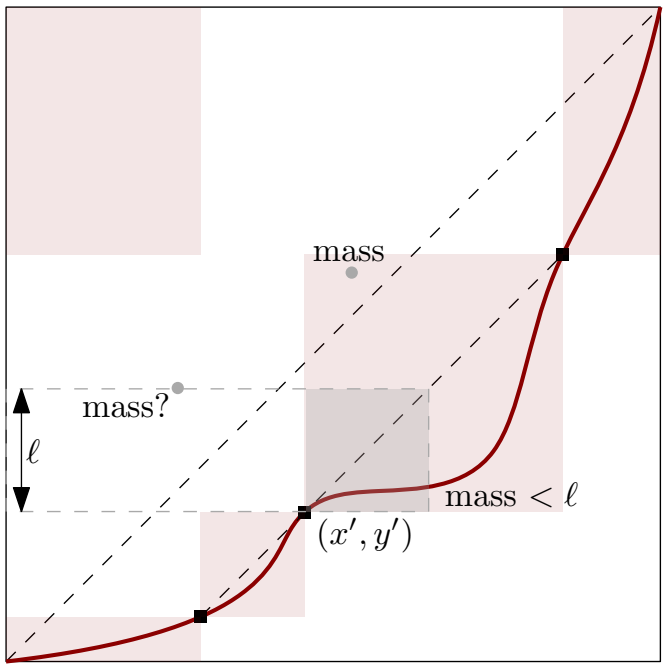

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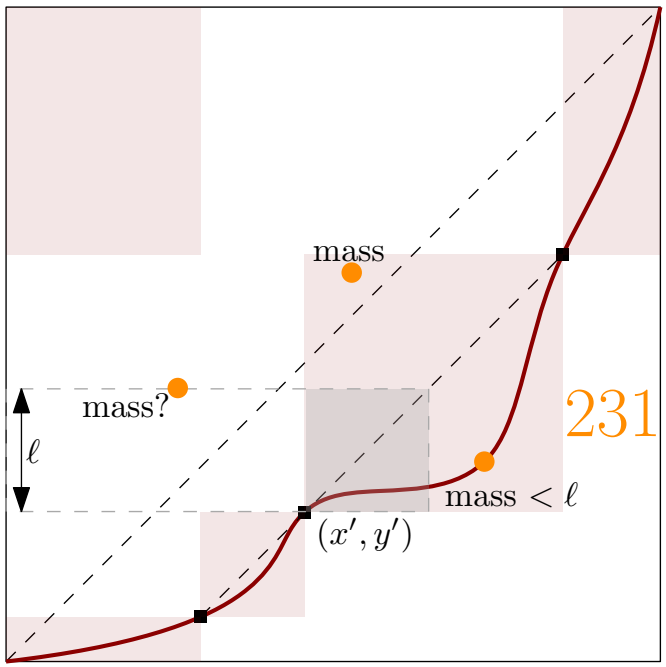

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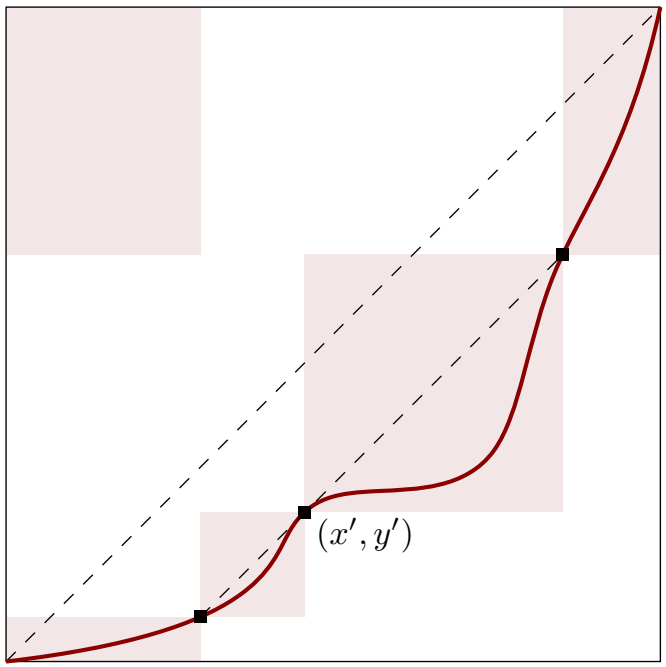

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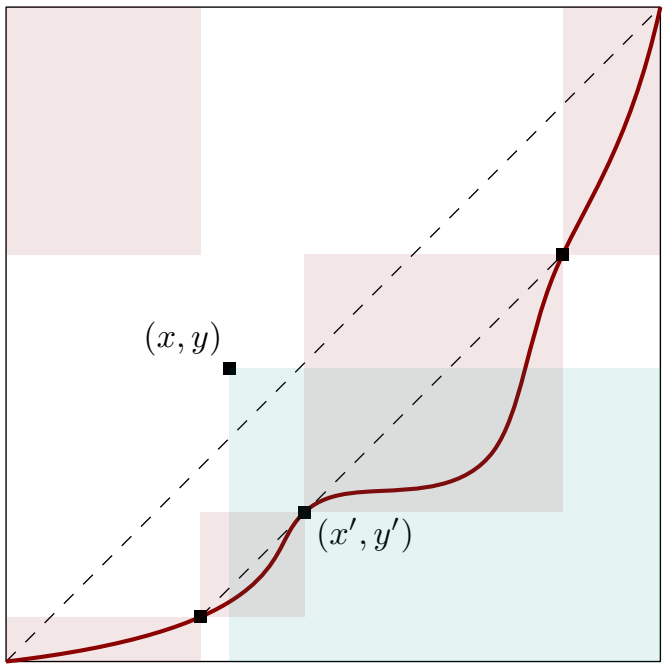

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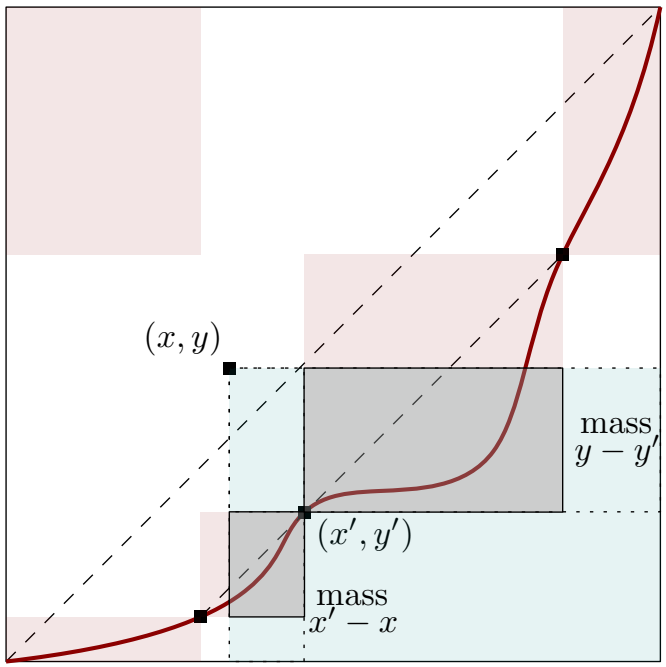

$$\mu([x,1] \times [0,y]) = y - y' + x' - x$$

LDP for uniform 231-avoiding permutations

[BDFKM'26]

Let $\sigma_n \sim \mathrm{Unif}(\mathrm{Av}_n(231))$. The RLM excursions φ_n satisfy an LDP in $\mathcal{E}$ at speed n with rate function

$$H^{231} : \varphi \in \mathcal{E} \mapsto \int_0^1 J(\varphi'(t))\mathrm{d}t$$

where $J(y) = (y+1)\log(y+1) + (1-y)\log(1-y)$.

Informally, this means that $\mathbb{P}(\varphi_n \approx \varphi) \approx \exp\big(-nH^{231}(\varphi)\big)$.

Proof idea: We know the LDP for simple random walks.

1. The proportion of Dyck paths among simple walks is polynomial, not exponential;
2. If w_n is a simple walk close to $\varphi \in \mathcal{E}$, it can be mapped to a nearby φ_n with not too much redundancy.

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Informally, this means that $\mathbb{P}(\varphi_n \approx \varphi) \approx \exp\big(-nH^{231}(\varphi)\big)$.

Proof idea: We know the LDP for simple random walks.

1. The proportion of Dyck paths among simple walks is polynomial, not exponential;
2. If w_n is a simple walk close to $\varphi \in \mathcal{E}$, it can be mapped to a nearby φ_n with not too much redundancy.

LDP for tilted 231-avoiding permutations

[BDFKM'26]

Let $\sigma_n \sim \mathrm{Unif}(\mathrm{Av}_n(231))$. The RLM excursions φ_n satisfy an LDP in $\mathcal{E}$ at speed n with rate function

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If $\tau_n \sim \mathrm{Mallows}(n, e^{\beta/n}, 231)$ then the associated φ_n has PMF

$$\mathbb{P}(\varphi_n = \varphi) \propto e^{2\beta n \int \varphi}$$

Application of Varadhan's lemma

Let $\tau_n \sim \mathrm{Mallows}(n, e^{\beta/n}, 231)$. The RLM excursions φ_n satisfy an LDP in $\mathcal{E}$ at speed n with rate function

$$\varphi \in \mathcal{E} \mapsto A_\beta(\varphi) - \inf A_\beta$$

where $A_\beta(\varphi) = H^{231}(\varphi) - 2\beta \int \varphi$.

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Finding the limit

Optimization [BDFKM'26]

A_β has a unique minimizer on $\mathcal{E}$

$$\varphi_\beta : t \in [0,1] \mapsto \frac{1}{\beta} \log\left(\frac{1+e^\beta}{1+e^{\beta(1-2t)}}\right) - t$$

Sketch of proof:

1. The space $\mathcal{E}$ and the function A_β are convex.
2. The directional derivatives of A_β vanish at φ_β.

Consequences: If $\tau_n \sim \mathrm{Mallows}(n, e^{\beta/n}, 231)$ then $\varphi_n \to \varphi_\beta$ in $\mathcal{E}$, therefore, $\mu_{\tau_n} \to \mu_{\varphi_\beta}$ in $\mathrm{Av}_\square(231)$.

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1 Pattern avoidance

2 Mallows models

3 Limit shapes

4 Strategy

5 The 231 case

6 The 321 case