Scaling limits of descent-biased trees

Victor Dubach

joint work with Paul Thévenin (Angers)
and Stephan Wagner (Graz)

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1 Model

2 Simulations

3 Real trees

4 Dendrons

5 Results

6 References

7 Proof ideas

Rooted non-plane labelled trees

$\mathbb{T}_n$: set of rooted non-plane labelled trees with n vertices
$\longleftrightarrow$ connected acyclic graphs on vertices labelled by $\{1, \dots, n\}$, with a distinguished one.

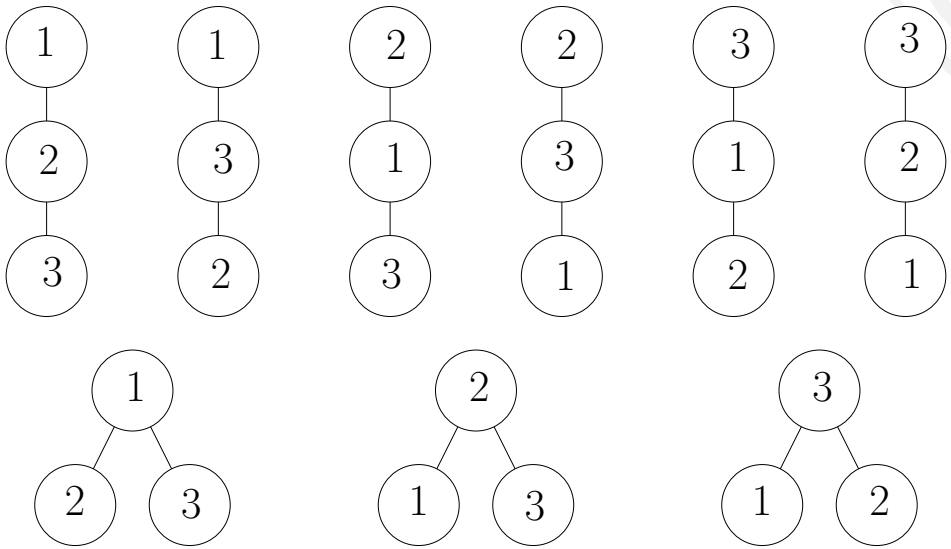

Figure 1: The 9 elements of $\mathbb{T}_3$.

Cayley's formula: $|\mathbb{T}_n| = n^{n-1}$.

Descents in rooted labelled trees

Let $t \in \mathbb{T}_n$. A *descent* is a couple (parent, child) where the label of the parent is greater than the label of the child.

$\mathrm{des}(t) :=$ number of descents in t.

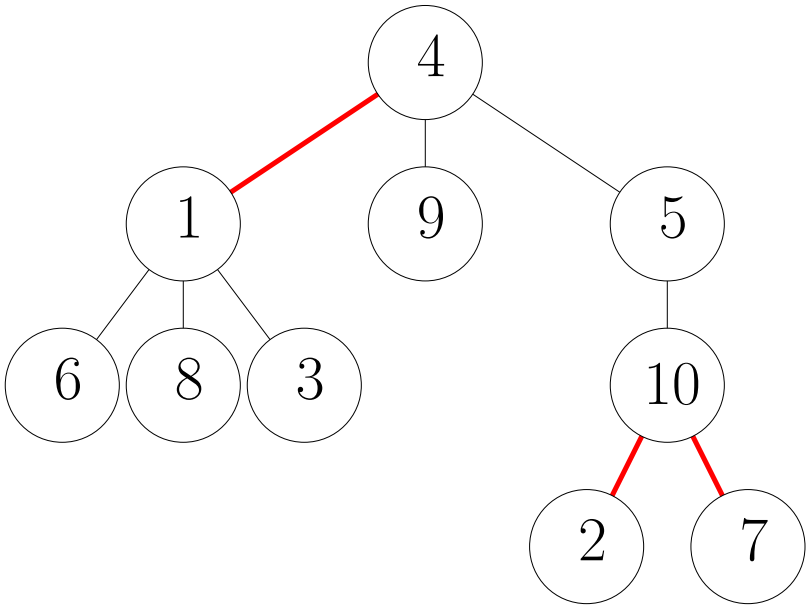

Figure 2: A tree of size 10 with 3 descents (in red)

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Special case: a tree with no descent is a *recursive tree*.

Enumeration: $\left|\mathbb{T}_n^0\right| = |\{t \in \mathbb{T}_n : \mathrm{des}(t) = 0\}| = (n-1)!$.

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Two standard probabilistic models:

- Uniform distribution on $\mathbb{T}_n$ (Cayley tree);
- Uniform distribution on $\mathbb{T}_n^0$ (random recursive tree).

Interpolation?

The q-descent-biased distribution

The model [Thévenin and Wagner, 2023]

Let $q \geq 0$. The *q-descent-biased tree* is the random variable $T_n^{(q)}$ with distribution:

$$\forall t \in \mathbb{T}_n, \quad \mathbb{P}\left(T_n^{(q)} = t\right) = \frac{1}{Z_{n,q}} q^{\mathrm{des}(t)}$$

where $Z_{n,q} = \sum_{t \in \mathbb{T}_n} q^{\mathrm{des}(t)}$.

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- We can restrict our attention to $q \in [0,1]$ thanks to the transformation $i \mapsto n - i + 1$ on labels.
- $Z_{n,q} = \prod_{k=1}^{n}(n - k + kq)$, [Eğecioğlu and Remmel, 1986].

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Local limit [Thévenin and Wagner, 2023]

For fixed $q \in (0, 1]$, the sequence of random trees $(T_n^{(q)})_{n \geq 1}$ admits a local limit at the root, which is one-ended.

Rem: no local limit if $q = q_n \to 0$.

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What about the *global* limit / scaling limit?

1 Model

2 Simulations

3 Real trees

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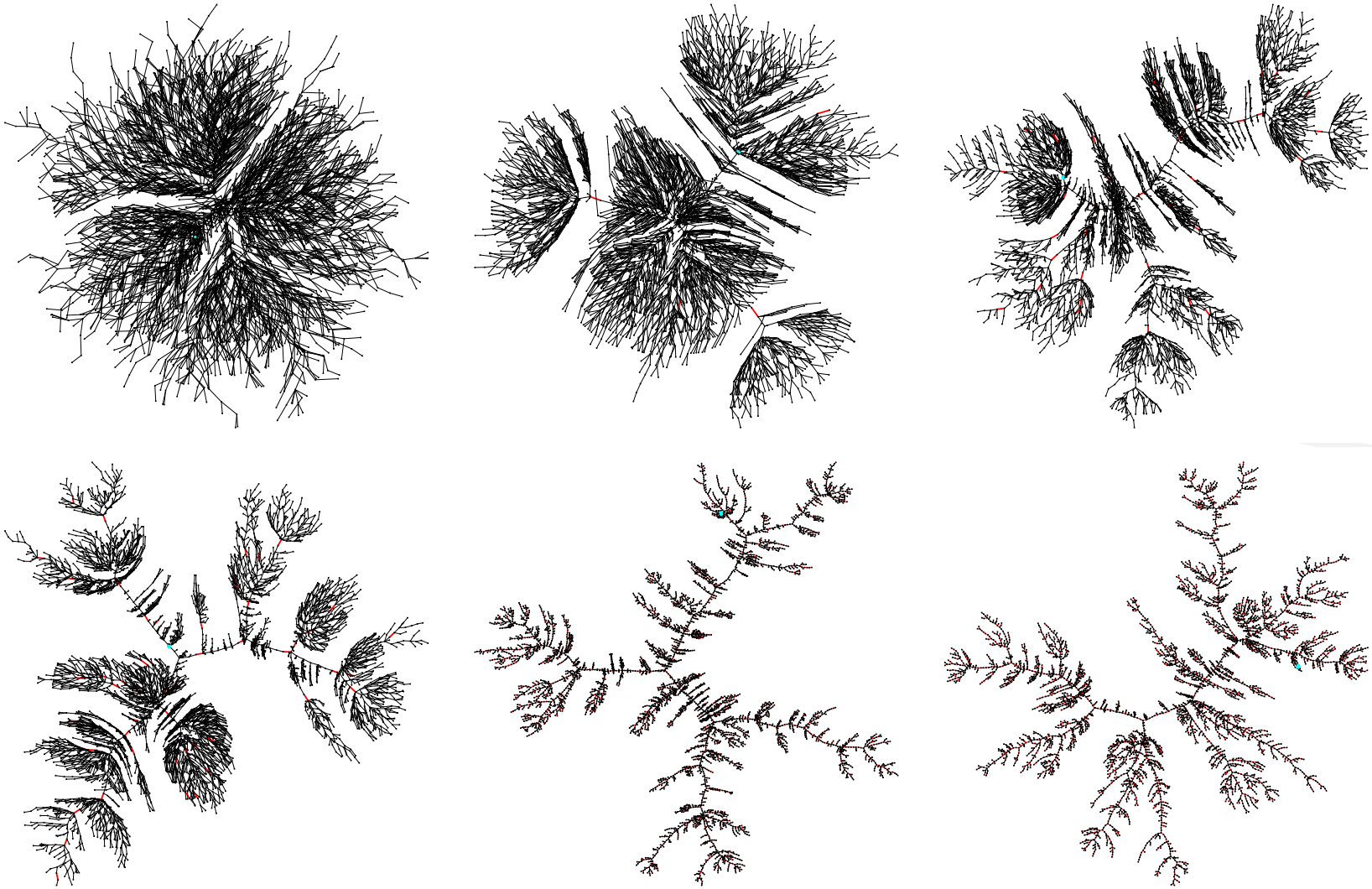
Simulations for $n = 5000$ and $q \in \{0,\ 1/n,\ 5/n,\ 10/n,\ 0.5,\ 1\}$



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4 Dendrons

5 Results

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Metric-measure spaces

A rooted mm-space $(\mathcal{X}, \rho, d, \mu)$ is a probability space $(\mathcal{X}, \mu)$ with a distinguished point $\rho \in \mathcal{X}$ and a measurable metric $d : \mathcal{X}^2 \to \mathbb{R}_+$.

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Ex 1: We view each tree $t \in \mathbb{T}_n$ as a rooted mm-space with d the graph distance and μ the uniform measure on vertices. If $\alpha > 0$, we denote by $\alpha \cdot t$ the rescaled mm-space where edges have length α.

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Ex 2: A real tree is a rooted mm-space $(\mathcal{T}, \rho, d, \mu)$ such that $(\mathcal{T}, d)$ is complete separable and the following holds for each $(u, v) \in \mathcal{T}^2$:

- There exists an isometric map $\gamma_{u,v} : [0, d(u, v)] \to \mathcal{T}$ with $\varphi(0) = u$ and $\varphi(d(u, v)) = v$. Its range is denoted by $[\![u, v]\!]$.
- If $\gamma : [0, 1] \mapsto \mathcal{T}$ is a continuous injective map with $\gamma(0) = u$ and $\gamma(1) = v$ then its range is $[\![u, v]\!]$.

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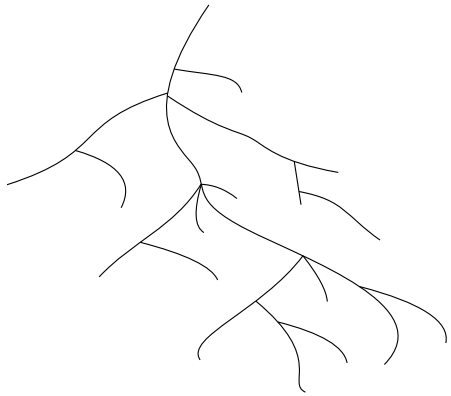

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Gromov-weak topology

Let $(\mathcal{X}_n, \rho_n, d_n, \mu_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be mm-spaces. We say that $\mathcal{X}_n$ converges towards $\mathcal{X}_\infty$ for the Gromov-weak topology if, for all $p \geq 1$:

$$\left(d_n\left(x_i^{(n)}, x_j^{(n)}\right)\right)_{0\leq i<j\leq p} \xrightarrow[n\to\infty]{\text{distrib}} \left(d_\infty\left(x_i^{(\infty)}, x_j^{(\infty)}\right)\right)_{0\leq i<j\leq p}$$

where, for each $n \in \mathbb{N}\cup\{\infty\}$, $(x_i^{(n)})_{i\geq 1}$ are i.i.d. μ_n-distributed random points and $x_0^{(n)} := \rho_n$.

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Case of interest: if $\mathcal{X}_n = \alpha_n \cdot \mathcal{T}_n$ and $\mathcal{X}_\infty = \mathcal{T}$, this corresponds to the convergence in distribution of the rescaled subtree spanned by p i.i.d. points.

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What about the random trees we are interested in?

The Brownian Continuum Random Tree

Convergence of Cayley trees [Aldous, 1993]

For each n, let T_n be a Cayley tree of size n. There exists a random real tree $\mathcal{T}_{\mathbf{e}}$, called the Brownian CRT, such that:

$$\frac{1}{\sqrt{n}} \cdot T_n \quad \xrightarrow[n\to\infty]{\text{distrib}} \quad \mathcal{T}_{\mathbf{e}}$$

for the Gromov-weak topology.

Figure 2: A Cayley tree of size 5000. Descents are in red, the root in blue.

Trees with logarithmic height

Distances in recursive trees [Panholzer, 2004]

For each n, let T_n be a random recursive tree. Let ρ_n denote the root, and let u_n, v_n be two i.i.d. random vertices. Then:

$$\begin{cases} d_n(u_n, \rho_n) = \log n + \mathcal{O}_{\mathbb{P}}(\sqrt{\log n}) \\ d_n(u_n, v_n) = 2\log n + \mathcal{O}_{\mathbb{P}}(\sqrt{\log n})\,. \end{cases}$$

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$\longrightarrow$ the Gromov-weak scaling limit would be a continuum of points at distance 1 from the root and 2 from each other... Not separable!

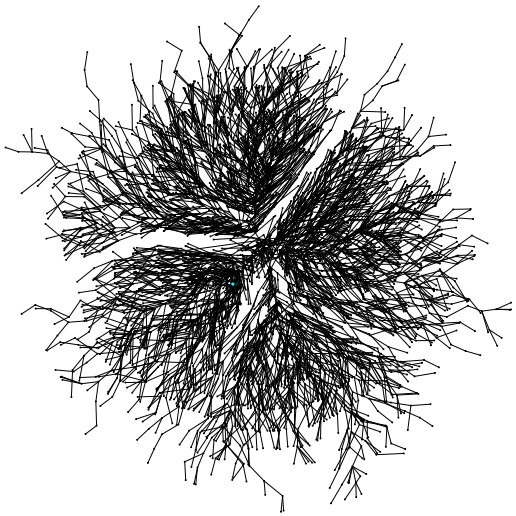


Figure 3: A random recursive tree of size 5000.

1 Model

2 Simulations

3 Real trees

4 Dendrons

5 Results

6 References

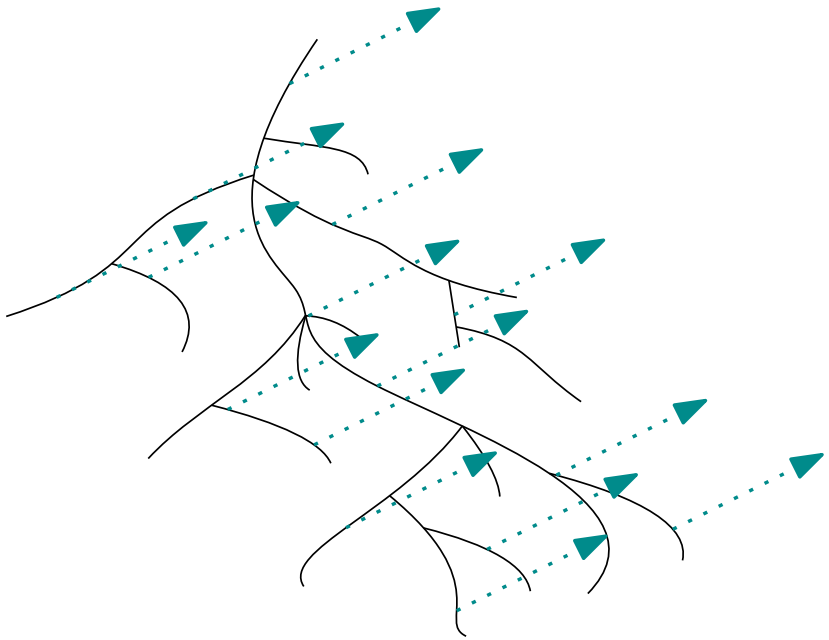
7 Proof ideas

Dendrons: definition

A dendron is a quadruple $\mathcal{D} = (\mathcal{T}, \rho_{\mathcal{T}}, d_{\mathcal{T}}, \nu)$ where $(\mathcal{T}, \rho_{\mathcal{T}}, d_{\mathcal{T}})$ is a real tree (with no measure) and ν is a Borel probability measure on $\mathcal{T} \times \mathbb{R}_+$.

The "distance" between two points $(u, a), (v, b) \in \mathcal{T} \times \mathbb{R}_+$ is:

$$d_{\mathcal{D}}\left((u, a), (v, b)\right) := a + d_{\mathcal{T}}(u, v) + b\,.$$



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Ex 2: Any probability measure m on $\mathbb{R}_+$ is mapped to a dendron Υ_m, with underlying real tree $\mathcal{T} := \{*\}$ and ν the pushforward of m by $a \mapsto (*, a)$.

Dendrons: topology

Let $(\alpha_n \cdot T_n)_{n\geq 1}$ be a sequence of rescaled trees, and let $\mathcal{D}$ be a dendron. We say that $\alpha_n \cdot T_n$ converges to $\mathcal{D}$ if, for all $p \geq 1$:

$$\left(\alpha_n \cdot d_n\left(x_i^{(n)}, x_j^{(n)}\right)\right)_{0\leq i<j\leq p} \xrightarrow[n\to\infty]{\text{distrib}} \left(d_{\mathcal{D}}\left(x_i, x_j\right)\right)_{0\leq i<j\leq p}$$

where $(x_i)_{i\geq 1}$ are ν-i.i.d. random points on $\mathcal{T} \times \mathbb{R}_+$, $x_0 := (\rho_{\mathcal{T}}, 0)$, and for each $n \geq 1$, $(x_i^{(n)})_{i\geq 1}$ are i.i.d. uniform random points on T_n and $x_0^{(n)} := \rho_n$.

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- The limit dendron is unique up to isomorphism;
- If the distance matrices of $\alpha_n \cdot T_n$ converge in distribution, then $\alpha_n \cdot T_n$ converges to some dendron;
- All dendrons are limits of rescaled finite trees.

Dendrons: non-trivial examples

[Bellin et al., 2025a]

Random trees built by uniform attachment with freezing converge in probability to a dendron Υ_{δ_c}, if the number of active vertices is linear (after log-rescaling);

[Bellin et al., 2025b]

If the number of active vertices is $\simeq n^\alpha$ for some $\alpha \in (\frac{1}{2}, 1)$, they converge to a dendron Υ_ν for some diffuse measure ν (after power-rescaling);

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[D., Thévenin, Wagner 2026+]
Random descent-biased trees in the critical regime converge in distribution to a non-trivial random dendron $\mathcal{D}_a$ (after log-rescaling).

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7 Proof ideas

Supercritical regime

Figure 4: Simulation of a descent-biased tree, $n = 5000$, $q = 0.5$.

Supercritical regime

Universality for fixed q [D., Thévenin, Wagner 26+]

Fix $q \in (0,1]$. For each $n \geq 1$, let $T_n^{(q)}$ be the q-descent-biased tree of size n. Then:

$$\frac{q-1}{\sqrt{q}\log q} \cdot \frac{1}{\sqrt{n}} \cdot T_n^{(q)} \quad \xrightarrow[n\to\infty]{\text{distrib}} \quad \mathcal{T}_{\mathbf{e}}$$

for the Gromov-weak topology, where $\mathcal{T}_{\mathbf{e}}$ is the Brownian CRT.

Supercritical regime

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Fix $q \in (0,1]$. For each $n \geq 1$, let $T_n^{(q)}$ be the q-descent-biased tree of size n. Then:

$$\frac{q-1}{\sqrt{q}\log q} \cdot \frac{1}{\sqrt{n}} \cdot T_n^{(q)} \quad \xrightarrow[n\to\infty]{\text{distrib}} \quad \mathcal{T}_{\mathbf{e}}$$

for the Gromov-weak topology, where $\mathcal{T}_{\mathbf{e}}$ is the Brownian CRT.

Expected to hold if $q = q_n$ is allowed to depend on n and $n \cdot q_n \to \infty$.

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Proof:
Analytic combinatorics, pumping method, singularity analysis...

Phase transition

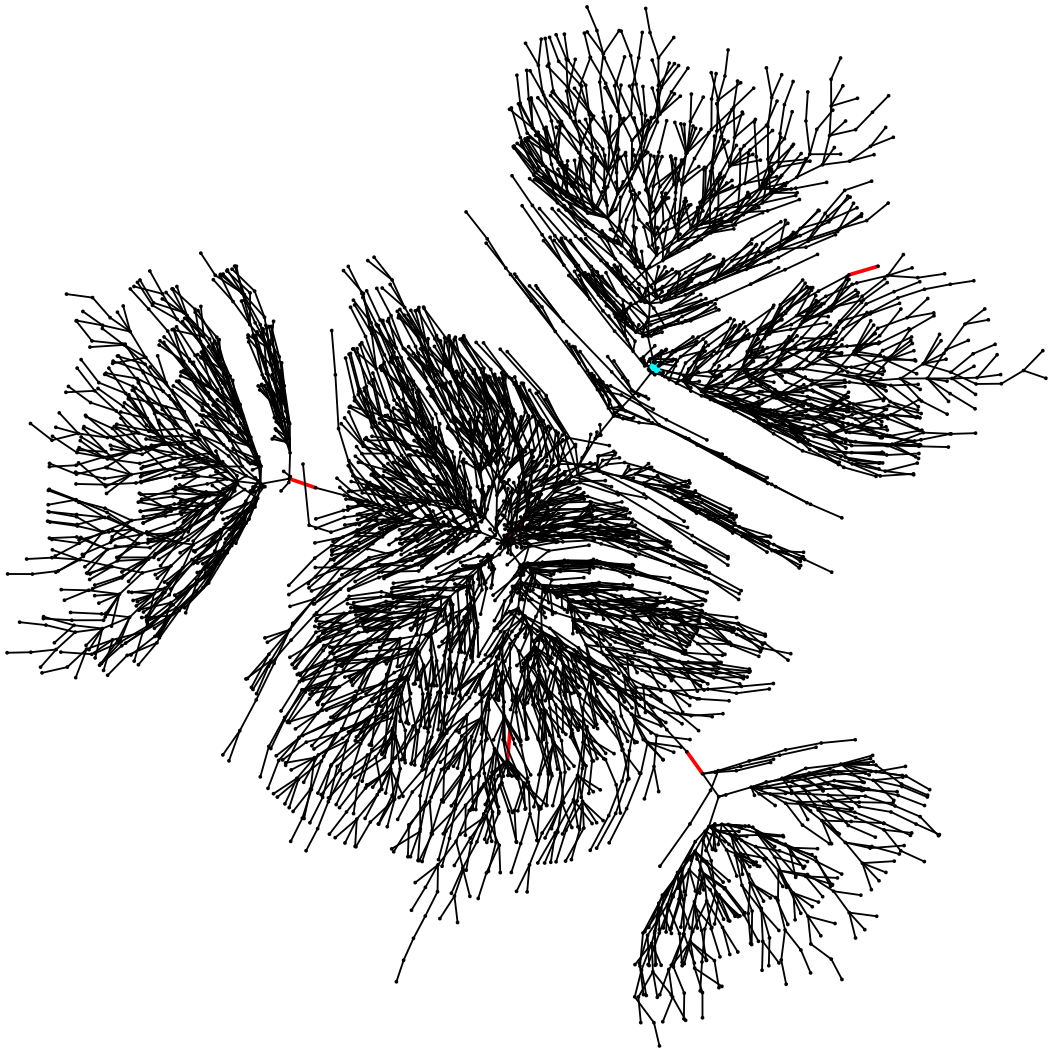
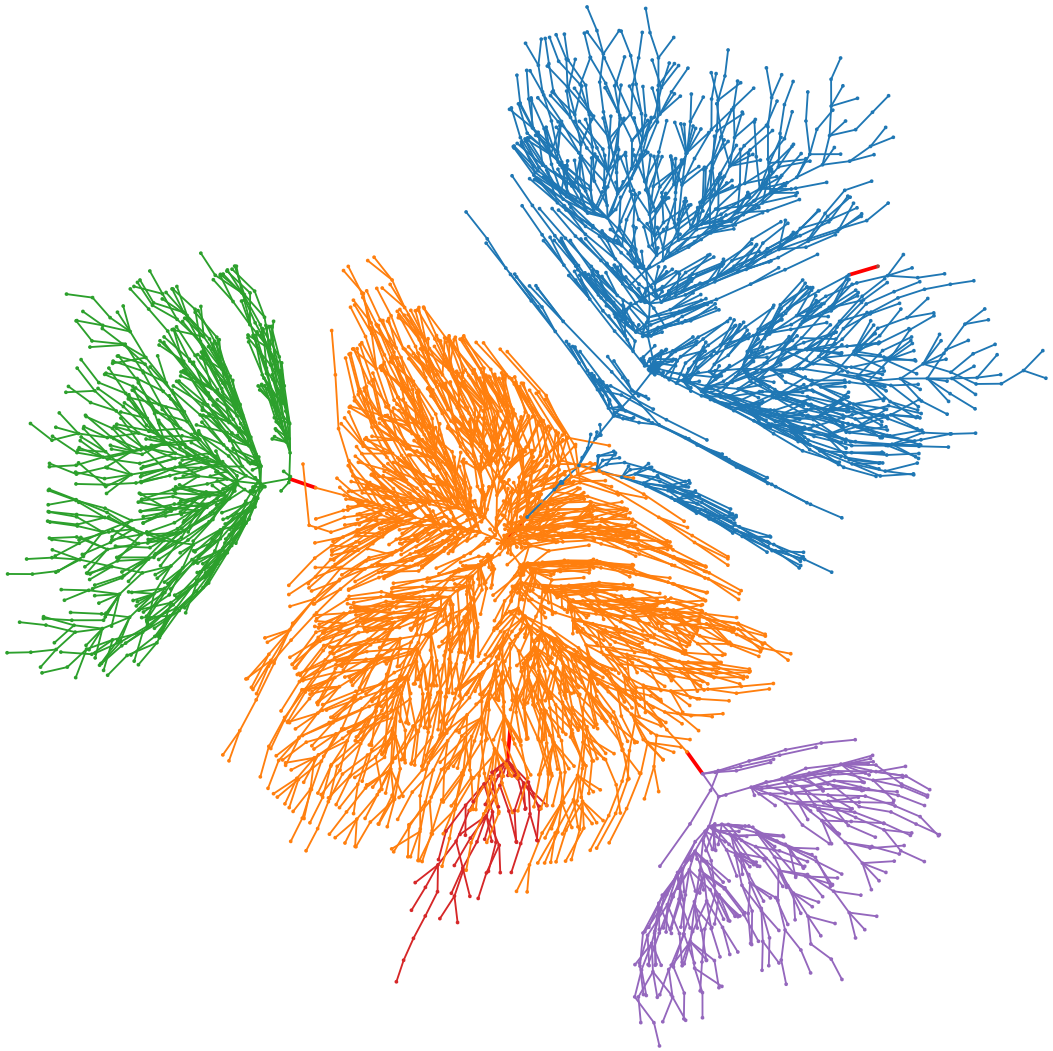


Figure 4: $n = 5000$, $q = 1/n$

Phase transition



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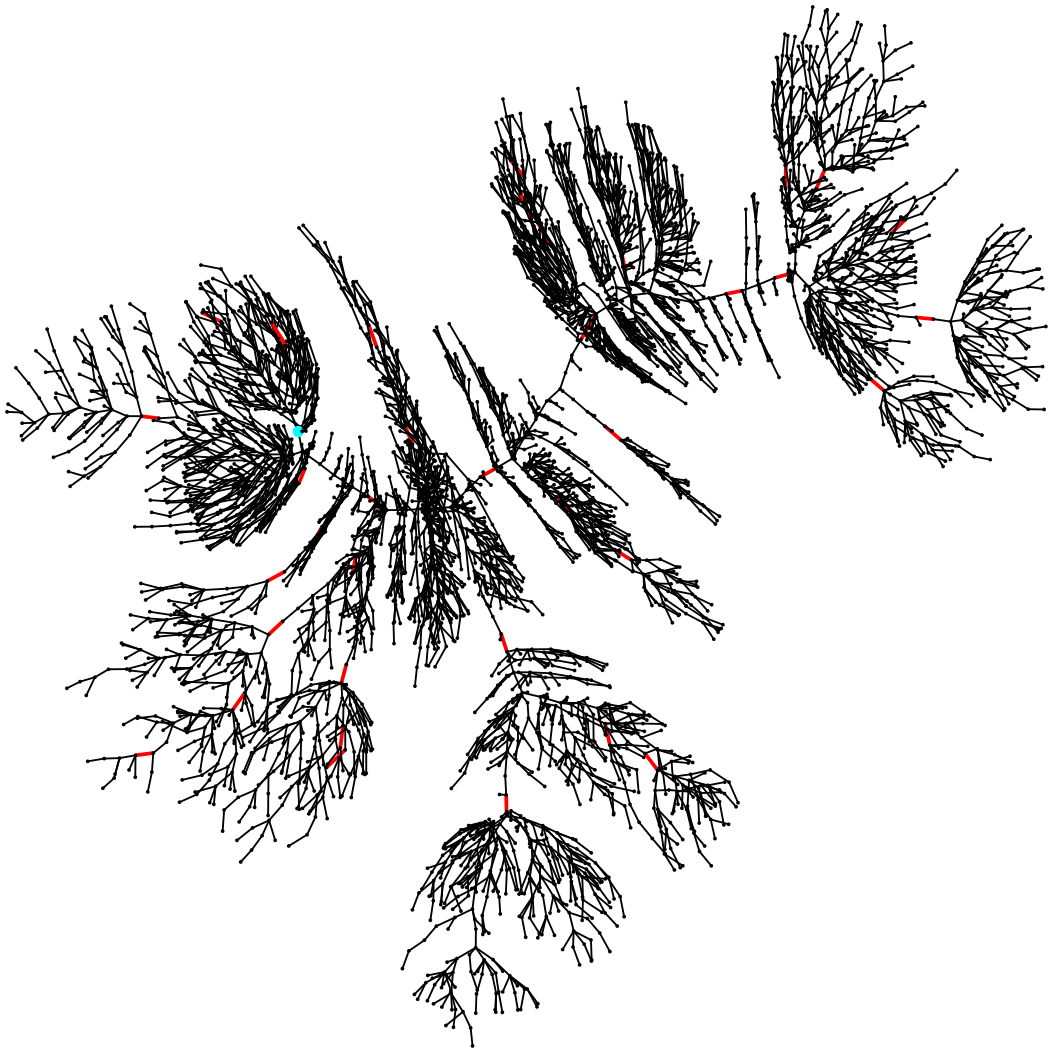

Figure 4: $n = 5000$, $q = 5/n$

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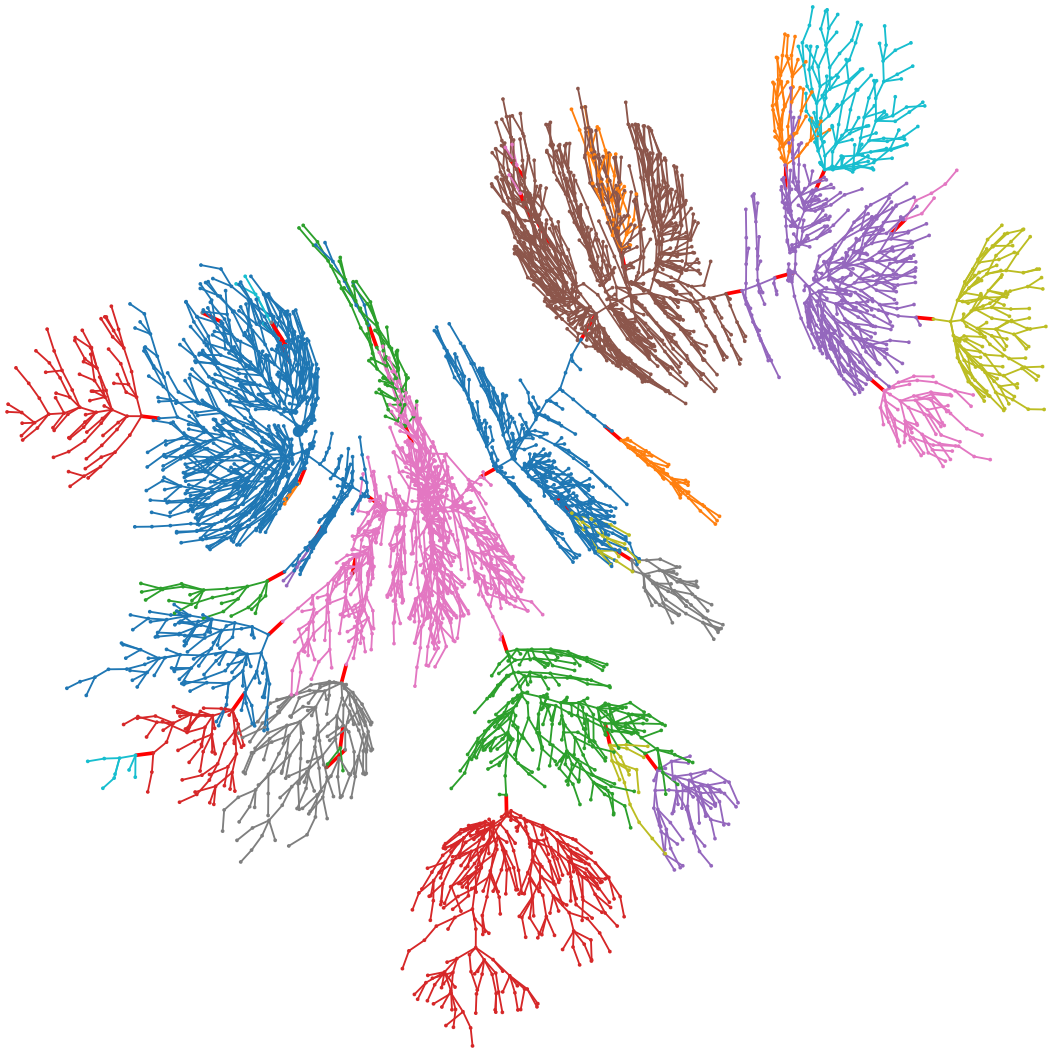

Figure 4: $n = 5000$, $q = 5/n$

Phase transition

Figure 4: $n = 5000,\ q = 10/n$

Phase transition

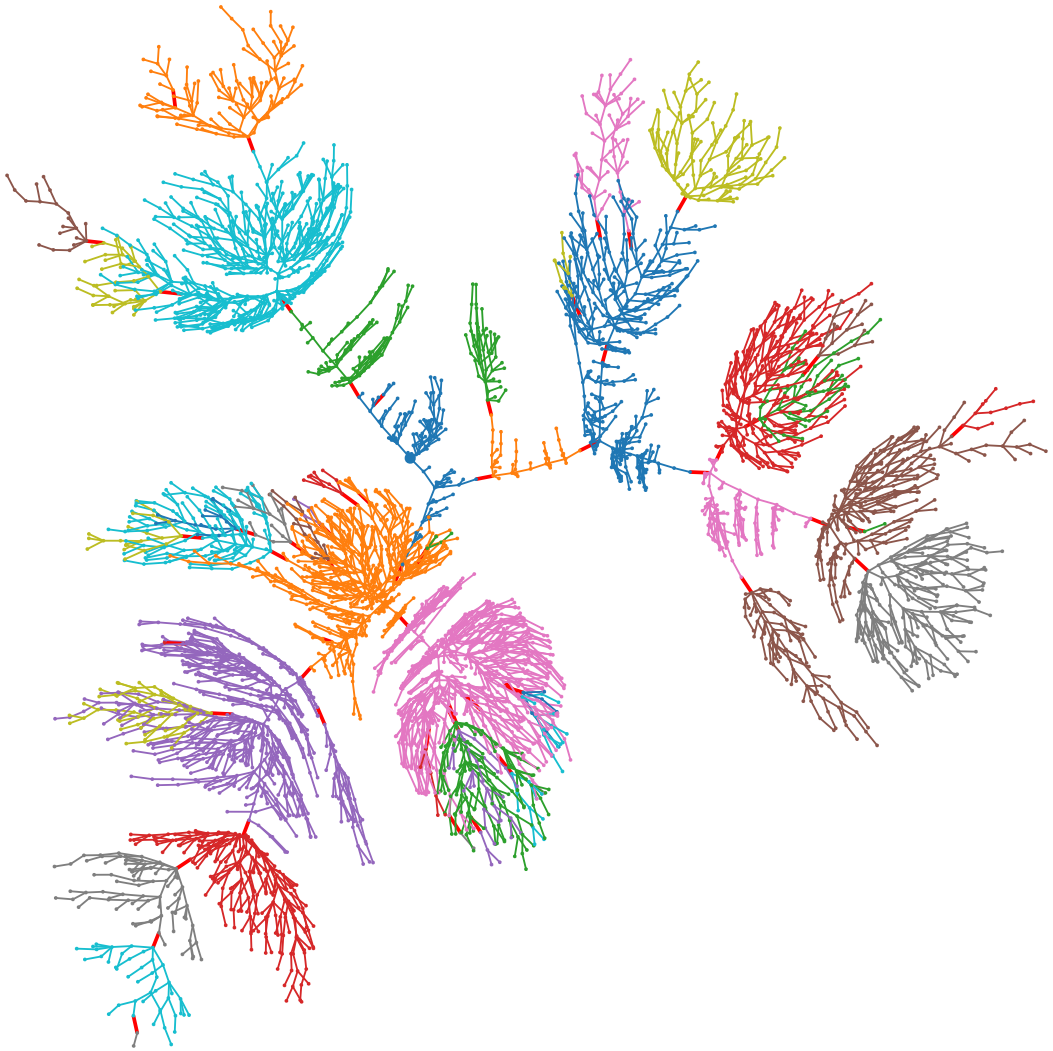

Figure 4: $n = 5000,\ q = 10/n$

Phase transition

Dendron limit for $q = a/n$ [D., Thévenin, Wagner 26+]

Fix $a > 0$. For each $n \geq 1$, let $q_n = a/n$ and let $T_n^{(q)}$ be the q_n-descent-biased tree of size n. Then:

$$\frac{1}{\log n} \cdot T_n^{(q)} \xrightarrow[n\to\infty]{\text{distrib}} \mathcal{D}_a$$

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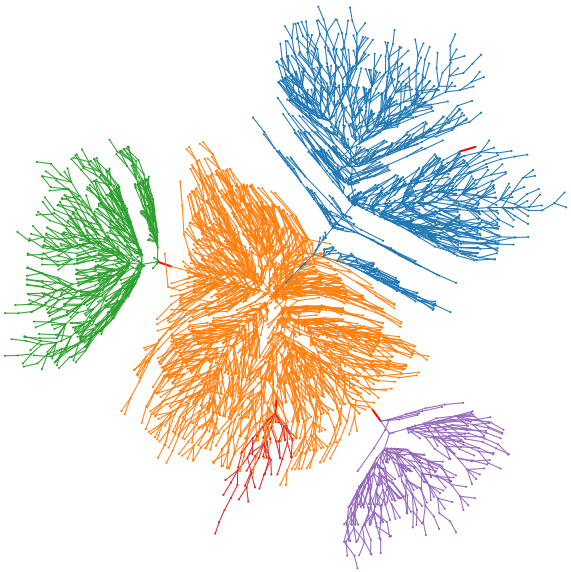

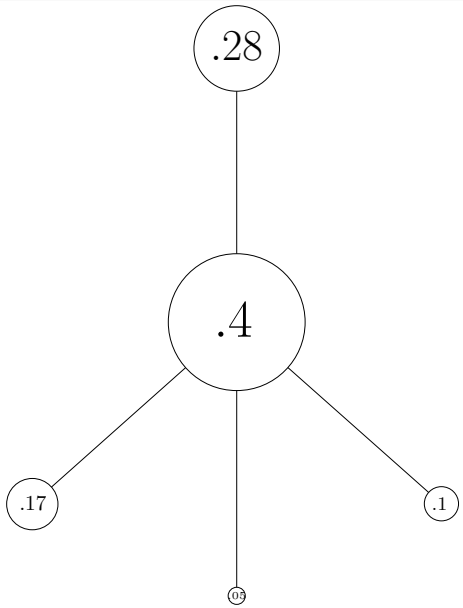

Figure 4: The limit dendron has a discrete base tree. Its edges have length 1 and its nodes u have weights α_u. The mass measure of the dendron is $\sum_u \alpha_u \, \delta_{(u,1)}$.

The limit dendron

First sample $B_1, B_2, \ldots$ i.i.d. $\mathrm{Beta}(1, a)$.
Then define $X_1 := B_1$, $X_2 = B_2(1 - X_1)$, $X_3 = B_3(1 - X_1 - X_2)$, ...

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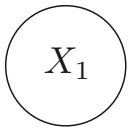

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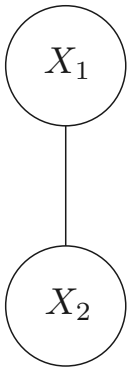

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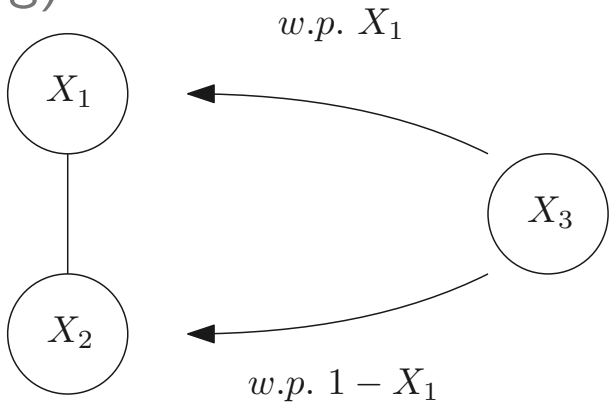

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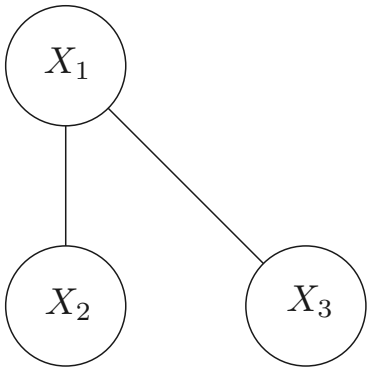

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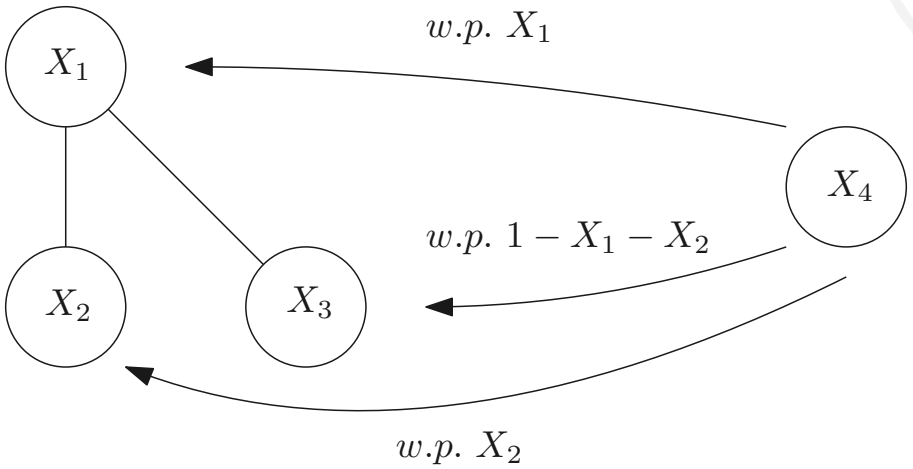

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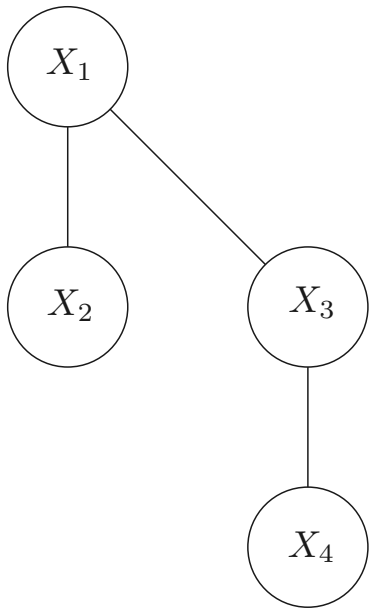

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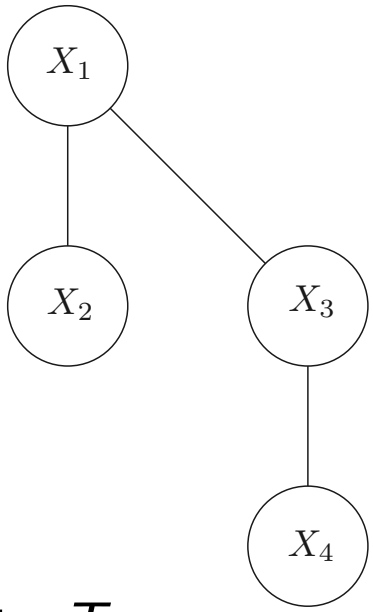

Adding $k+1$ to T_k:
choose $i \in [k-1]$ w.p. X_i, choose k w.p. $1 - \sum_{i \le k-1} X_i$.

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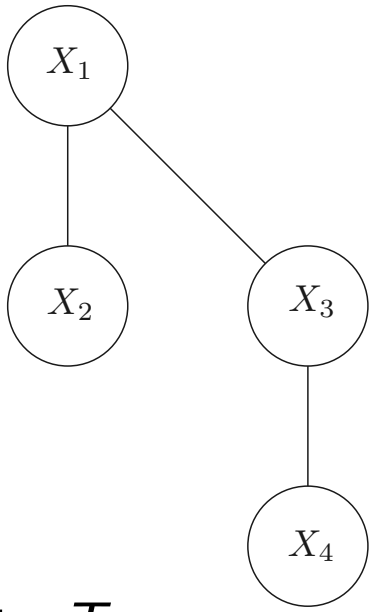

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Intuition:
the reduced tree structure is uniform conditionally on its degrees

1 Model

2 Simulations

3 Real trees

4 Dendrons

5 Results

6 References

7 Proof ideas

References I

Back to start

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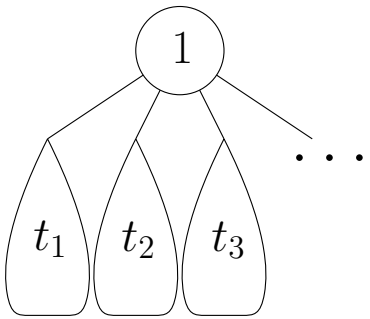
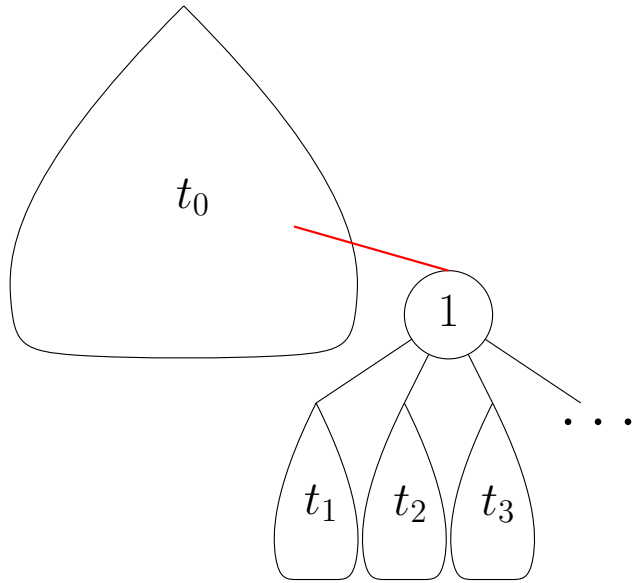
A decomposition to get differential equations

We can decompose a descent-biased tree t according to the location of the label 1:

- if the root has label 1, then t is a root + a family of descent-biased subtrees $t_1, \ldots, t_p$;



- otherwise, t is a descent-biased tree t_0 + a descent-edge leading to 1 + a family of descent-biased subtrees $t_1, \ldots, t_p$.



Generating function when q is constant

Consider the generating function

$$A(x,q) := \sum_{t\in\mathbb{T}} q^{\mathrm{des}(t)} \frac{x^{|t|}}{|t|!}\,.$$

Our decomposition yields $\partial_x A = e^A + qxe^A\partial_x A$, with implicit solution

$$x = \frac{e^{-qA} - e^{-A}}{1-q}\,.$$

We deduce that $A(x,q)$ is amenable to singularity analysis with a square root singularity at $x_0 = q^{q/(1-q)}$:

$$A(x,q) = \frac{\log q}{q-1} - \sqrt{\frac{2}{q}}\cdot\sqrt{1-x/x_0} + \mathcal{O}(|x-x_0|)\,.$$

By singularity analysis: $[x^n]A(x,q) \sim \frac{1}{\sqrt{2\pi q}} n^{-3/2} x_0^{-n}$.

Typical depth when q is constant

Now, depth of a typical vertex:

$$F(x,y,q) := \sum_{t\in\mathbb{T}} q^{\mathrm{des}(t)} \frac{x^{|t|}}{|t|!} \sum_{w\in t} y^{\mathrm{depth}(w)}\,.$$

Our decomposition yields a differential equation solved by

$$F = \frac{e^{(1-q)yA} - 1}{y - qye^{(1-q)yA}}\,.$$

Then, recursively pump the moments $\mathbb{E}[\mathrm{depth}(W_n)^r] \simeq \frac{[x^n]f_r(x,q)}{n[x^n]A(x,q)}$ through

$$f_r(x,q) := \partial_y^r F|_{y=1} \sim \frac{c}{\sqrt{q}} \left(\frac{\sqrt{q}\log q}{q-1}\right)^r (1-x/x_0)^{-1/2-r/2}\,.$$

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Finally, generalize to pairwise distances between multiple vertices.

Discrete asymptotics when $q = a/n$

In the regime $q = a/n$, we can still work out asymptotics with the right contour integration:

$$[x^n]A(x, a/n) \sim \frac{n^{a-1}}{e^{a-1}\Gamma(a+1)}.$$

Depth of typical vertex:

$$[x^n]F(x, y^{1/\log n}, q) \sim n^a e^{-a} \sum_{k\geq 1} k a^{k-1} y^k \frac{1}{\Gamma(a+k+1)},$$

implying that $\frac{1}{\log n}\mathrm{depth}(W_n)$ converges in distribution to a random variable on $\mathbb{N}^*$ (!)

To get the structure of the whole tree, we study the locations of the k smallest labels, for some $k = k_n$ that grows (slowly) with n. This describes the base tree of the dendron $\mathcal{D}_a$...