A geometric approach to conjugation-invariant permutations

Victor Dubach
under the supervision of Valentin Féray

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Outline

I. Introduction: conjugation-invariance

II. Tool: a geometric construction

III. Application: number of records

I. Conjugation-invariance

Definition

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Property

τ is conjugation-invariant

$\Longleftrightarrow$

conditionally given t, τ is a uniformly random t-cyclic permutation.

$\longrightarrow$ we may study uniform permutations with given cycle types.

I. Conjugation-invariance

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- Uniform involutions with no fixed points
  (deterministic cycle type with $2t_2 = n$)

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  (random cycle type with $t_1 + 2t_2 = n$ and $t_1 \approx \sqrt{n}$)
- Uniform involutions with no fixed points
  (deterministic cycle type with $2t_2 = n$)
- Ewens random permutations

$$\mathrm{P}_\theta(\tau) \propto \theta^{\sum t_p}$$

and their generalizations.

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Here: a geometric approach that works for all cycle types, under minimal hypotheses.

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II. Tool: a geometric construction

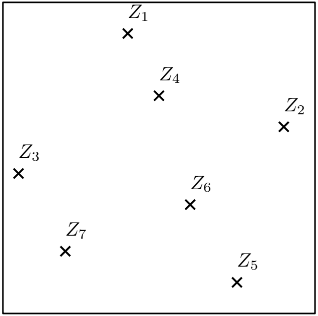
III. Application: number of records

II. A geometric construction

Mapping a point set to a permutation

Let $\mathcal{Z} = \{Z_1 = (X_1, Y_1), \ldots, Z_n = (X_n, Y_n)\} \subset [0,1]^2$ with distinct x-coordinates and distinct y-coordinates. Define $\tau = \mathrm{Perm}(\mathcal{Z})$ by:

$\tau(i) = j$ iff the i-th point from the left is the j-th point from the bottom.



$\mathrm{Perm}(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7) = 4\ 2\ 7\ 6\ 3\ 1\ 5$

II. A geometric construction

Recovering uniform permutations (e.g. Hammersley '70)

Let $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ be i.i.d. $\mathrm{Unif}\left([0,1]^2\right)$. Then $\tau = \mathrm{Perm}(\mathcal{Z})$ is a uniform permutation of $\{1, \ldots, n\}$.

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Recovering random involutions (Baik and Rains '99)

Let $\mathcal{Z}^*$ be the symmetry of $\mathcal{Z}$ with respect to the diagonal of $[0,1]^2$. Then $\tau = \mathrm{Perm}\left(\mathcal{Z} \cup \mathcal{Z}^*\right)$ is a uniform involution of $\{1, \ldots, 2n\}$ with no fixed point.

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Idea: the set $\mathcal{Z} \cup \mathcal{Z}^*$ is symmetric w.r.t. the diagonal, and has the same x-coordinates as y-coordinates.
If (X, Y) has relative position (i, j) in this set, then (Y, X) has relative position (j, i). Thus $\tau(i) = j$ and $\tau(j) = i$.

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Let $(U_i)_{1 \leq i \leq n}$ be i.i.d. $\mathrm{Unif}([0,1])$. For each i, set $Z_i := \big(U_i, U_{\mathfrak{s}(i)}\big)$. Then $\tau = \mathrm{Perm}(Z_i, 1 \leq i \leq n)$ is a uniform t-cyclic permutation.

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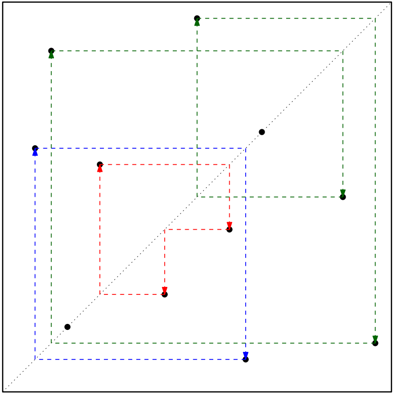

Geometric construction of a $(2,1,1,1)$-cyclic permutation.

$$\tau = 8\ 10\ 3\ 7\ 4\ 11\ 5\ 1\ 9\ 6\ 2$$
$$= (3)\circ(9)\circ(1,8)\circ(4,7,5)\circ(2,10,6,11)$$

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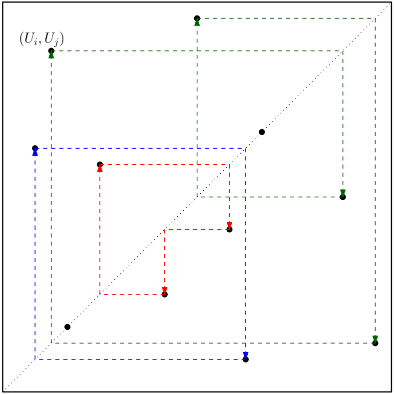

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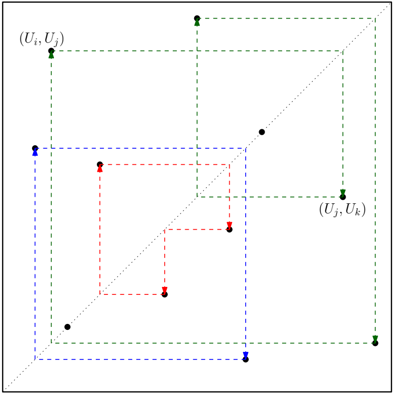

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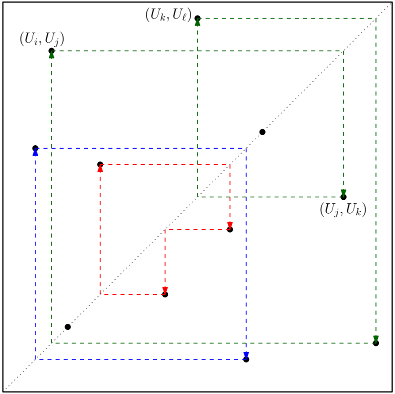

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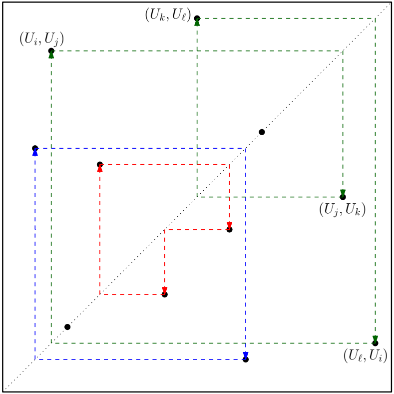

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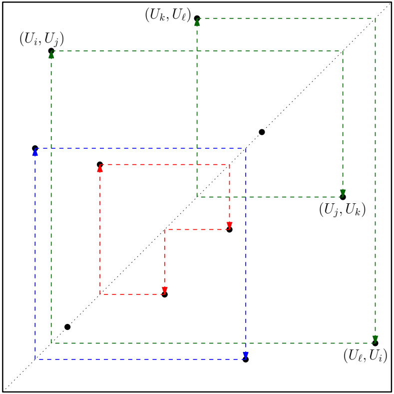

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Property: if R is a rectangle which does not intersect the diagonal, it contains $\sim n\cdot\mathrm{Leb}(R)$ i.i.d. uniform points.

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Theorem

If τ_n is a uniform permutation of size n, then

$$\frac{\mathrm{rec}(\tau_n) - \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) .$$

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Q: uniform permutations in given conjugacy classes ?

III. Application: number of records

If $\tau = \mathrm{Perm}(\mathcal{Z})$, a point $(X, Y) \in \mathcal{Z}$ corresponds to a record in τ iff there is no other point in its up-left corner $[0, X] \times [Y, 1]$.

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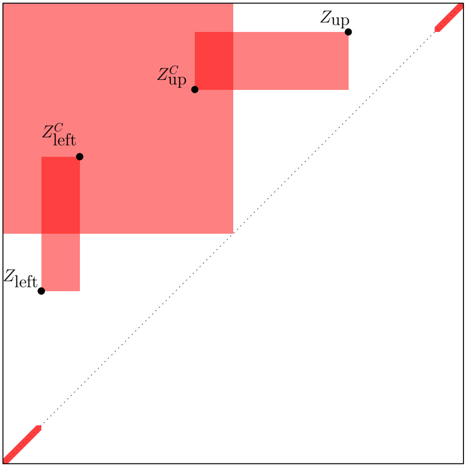

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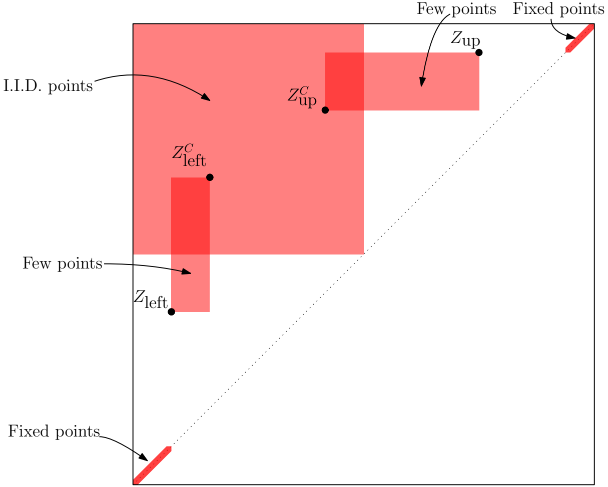

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Records of conjugation-invariant permutations (D. '24)

Let τ_n be conjugation-invariant permutations of size n, with (random) cycle types $t^{(n)}$. Write $\check{n} := n - t_1^{(n)}$. If $\check{n} \gg \frac{n}{\sqrt{\log n}}$ then:

$$\frac{\mathrm{rec}\left(\tau_n\right) - \log n}{\sqrt{\log n}} \longrightarrow \mathcal{N}\left(0, 1\right).$$

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More generally if $\check{n} \to \infty$ and $\frac{\check{n}}{n/\sqrt{\log n}} \to \alpha \in [0, \infty]$ in probability:

$$\frac{\mathrm{rec}\left(\tau_n\right) - \log\left(\check{n}\right)}{t_1^{(n)}/\check{n} + \sqrt{\log\left(\check{n}\right)}} \longrightarrow \frac{\alpha}{\alpha+1} Y + \frac{1}{\alpha+1}\Gamma_2$$

where Y and Γ_2 are independent $\mathcal{N}\left(0,1\right)$ and $\mathrm{Gamma}\left(2,1\right)$ r.v.'s.

Conclusion

The geometric construction can also be used to prove universality results for:

- Longest monotone subsequences (first order and concentration inequalities);
- Robinson–Schensted shapes;
- Pattern counts (asymptotic normality with explicit variance, concentration inequalities, and bounds on the speed of convergence).

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Thank you for your attention!

Bonus content

Longest monotone subsequences

$\forall \delta > 0, \exists c_\delta > 0$ s.t. for any conjugation-invariant τ_n:

$$\begin{cases} \mathbb{P}\left(\mathrm{LDS}(\tau_n) < (2-\delta)\sqrt{\check{n}}\right) \leq \mathbb{E}\left[\exp\left(-c_\delta \check{n}\right)\right]; \\ \mathbb{P}\left(\mathrm{LDS}(\tau_n) > (2+\delta)\sqrt{\check{n}}\right) \leq \mathbb{E}\left[\exp\left(-c_\delta \sqrt{\check{n}}\right)\right]. \end{cases}$$

Bonus content

Robinson–Schensted shape

For any conjugation-invariant τ_n, if $\check{n} \to \infty$:

$$\frac{1}{\check{n}} \mathrm{LDS}_{r\sqrt{\check{n}}}(\tau_n) \underset{n\to\infty}{\longrightarrow} F_{\text{Vershik-Kerov-Logan-Shepp}}(r)$$

in probability, for all $r \geq 0$.

Bonus content

Pattern counts

If $t_1^{(n)} = np_1 + o_{\mathrm{P}}(\sqrt{n})$ and $2t_2^{(n)} = np_2 + o_{\mathrm{P}}(n)$:

$$\left(\frac{\mathrm{Occ}_\pi(\tau_n) - \binom{n}{r}\mu_\pi^{p_1}}{n^{r-1/2}}\right)_{\pi\in\mathfrak{S}_r} \underset{n\to\infty}{\longrightarrow} \mathcal{N}\left(0, \Sigma^{p_1,p_2}\right)$$

for some "explicit" $\mu_\pi^{p_1}$ and $\left(\Sigma_{\pi,\rho}^{p_1,p_2}\right)_{\pi,\rho\in\mathfrak{S}_r}$.

- $\mu_\pi^0 = 1/r!$ and $\Sigma_{\pi,\pi}^{0,p_2} > 0$
- **Féray–Kammoun**: if $p_1 < 1$ then $\Sigma_{\pi,\pi}^{p_1,p_2} > 0$.
- Σ^{0,p_2} has rank $(r-1)^2$ if $p_2 < 1$, and rank $r(r-1)/2$ if $p_2 = 1$
- $d_K\left(\frac{\mathrm{Occ}_\pi(\tau_n)-\mathbb{E}[\mathrm{Occ}_\pi(\tau_n)]}{\sqrt{\mathrm{Var}[\mathrm{Occ}_\pi(\tau_n)]}}, \mathcal{N}\left(0,\Sigma_{\pi,\pi}^{p_1,p_2}\right)\right) = \mathcal{O}(n^{-1/2})$ if $p_1 < 1$
- $\mathbb{P}\left(\left|X_\pi(\tau_n) - \mathbb{E}\left[X_\pi(\tau_n)\right]\right| \geq t\right) \leq 2\exp\left(\frac{-2r!}{3}\frac{t^2}{n^r}\right)$