On Some Models of Non-Uniform Random Permutations

Victor Dubach

Ph.D. thesis defense, IECL Nancy

Outline

I. Models of non-uniform random permutations

II. A geometric construction

III. Binary search trees of permuton samples

IV. Pattern counts of conjugation-invariant permutations

V. Conclusion

I. Non-uniform random permutations

Goal:

Properties of **random** permutations

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Ex: number of fixed points, cycle lengths, ...

Ex: The uniform distribution $\mathrm{Unif}\left(\mathfrak{S}_n\right)$

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Ex: For $q \geq 0$, the Mallows distribution $\mathcal{M}_{n,q}$:

$$\forall \sigma \in \mathfrak{S}_n,\ \mathcal{M}_{n,q}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$$

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I. Non-uniform random permutations

	Uniform	Mallows
LIS/LDS	[VK'77, BDJ'99]	[MS'11, BP'15, BB'17]
RS	[VK'77, LS'77]	?
Patterns	[JNZ'15]	
BST	[D'86]	[ABC'21]
Records	Folklore	"easy"

	Conjugation-invariant	Permuton samples
LIS/LDS	[K'18]	[DZ'99]
RS	[K'18]	[S'22]
Patterns	[HR'22, FK'23]	"easy"
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LIS: Longest Increasing Subsequence; LDS: Longest Decreasing Subsequence;
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II. A geometric construction

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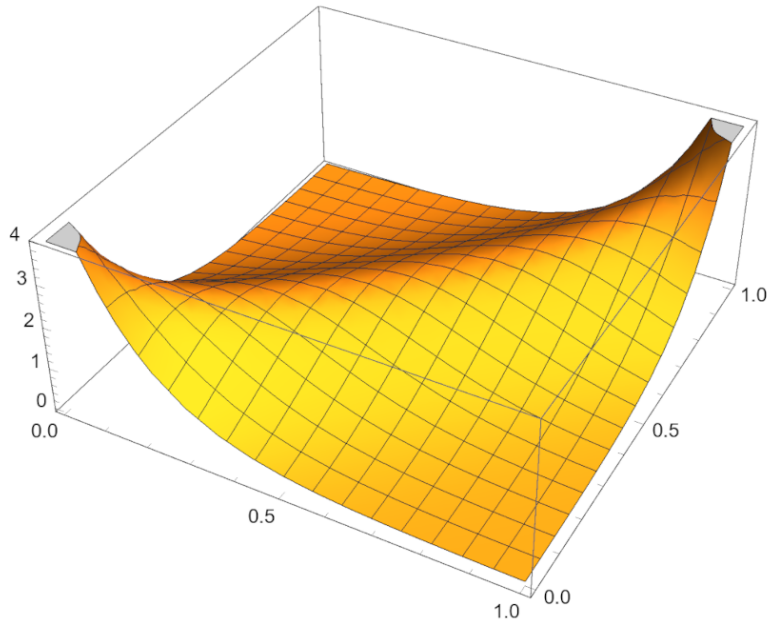
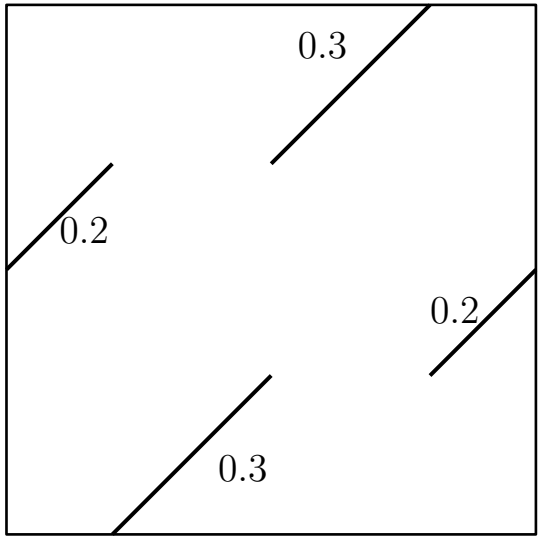
II. A geometric construction

Permutons

A *permuton* is a probability measure μ on $[0,1]^2$ with uniform marginals:

$$\forall t \in [0,1], \quad \mu([0,t]\times[0,1]) = \mu([0,1]\times[0,t]) = t.$$





II. A geometric construction

Mapping a point set to a permutation

Let $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{R}^2$ with no common x- or y-coordinate.

Define a reordering $(x_{(1)}, y_{(1)}), \ldots, (x_{(n)}, y_{(n)})$ by $x_{(1)} < \cdots < x_{(n)}$.

There exists a unique permutation $\sigma = \mathrm{Perm}(\mathcal{P})$ such that $(y_{(1)}, \ldots, y_{(n)})$ and $(\sigma(1), \ldots, \sigma(n))$ are in the same relative order.

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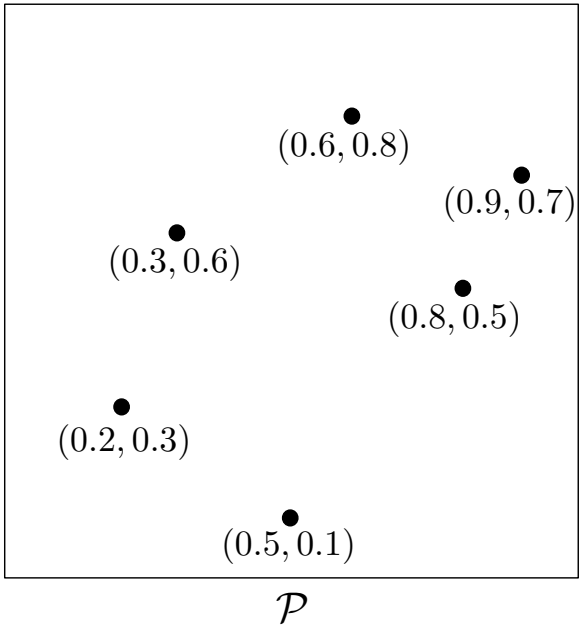


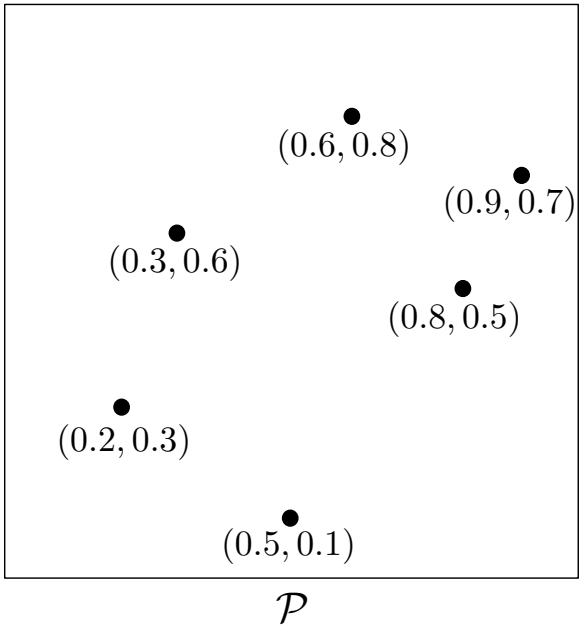
$$(y_{(1)}, \ldots, y_{(6)}) = (0.3, 0.6, 0.1, 0.8, 0.5, 0.7)$$

$$\mathrm{Perm}(\mathcal{P}) = (2, 4, 1, 6, 3, 5)$$

II. A geometric construction

From permutons to permutations

If $\mathcal{P}$ is random i.i.d. under a permuton μ, this permutation σ_μ^n is a *permuton sample.*



$\mathcal{P}$

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Rem: if $\mu = \mathrm{Leb}_{[0,1]^2}$ then σ_μ^n is uniformly random.

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- Continuous deformation of the uniform distribution

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- Continuous deformation of the uniform distribution
- Non-parametric, wide range of distributions
- "Asymptotic density": permuton samples can "approximate" any "regular" sequence of permutations...

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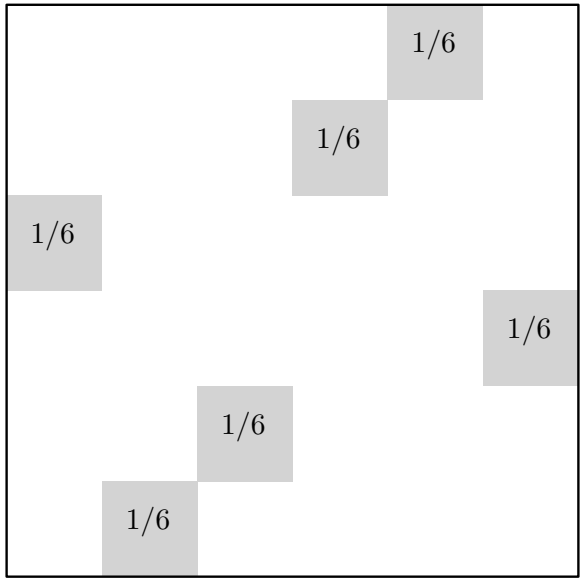
From permutations to permutons

To each permutation $\sigma \in S_n$ we can associate a permuton μ_σ with:

$$\frac{d\mu_\sigma}{d\mathrm{Leb}_{[0,1]^2}} = n \cdot \sum_{i=1}^{n} 1_{\left[\frac{i-1}{n},\frac{i}{n}\right]\times\left[\frac{\sigma(i)-1}{n},\frac{\sigma(i)}{n}\right]}.$$

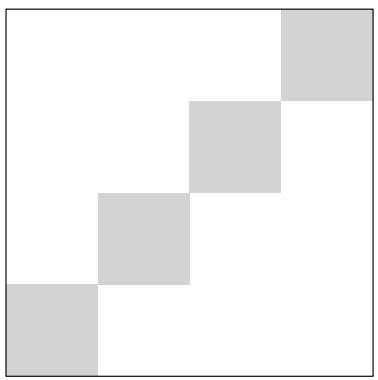
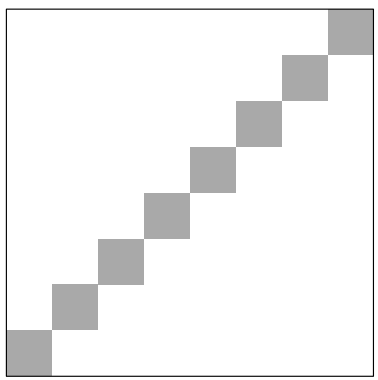
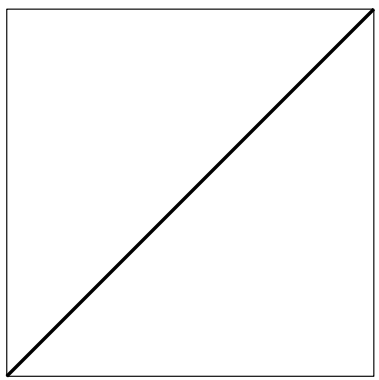
$\sigma = 4\ 1\ 2\ 5\ 6\ 3$ $\longrightarrow$



II. A geometric construction

Convergence of a sequence of permutations

If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of permutations and μ is a permuton, we say that $(\sigma_n)_{n\in\mathbb{N}}$ converges towards μ when $\mu_{\sigma_n} \underset{n\to\infty}{\longrightarrow} \mu$ weakly.

Identity permutations converge to the diagonal permuton.

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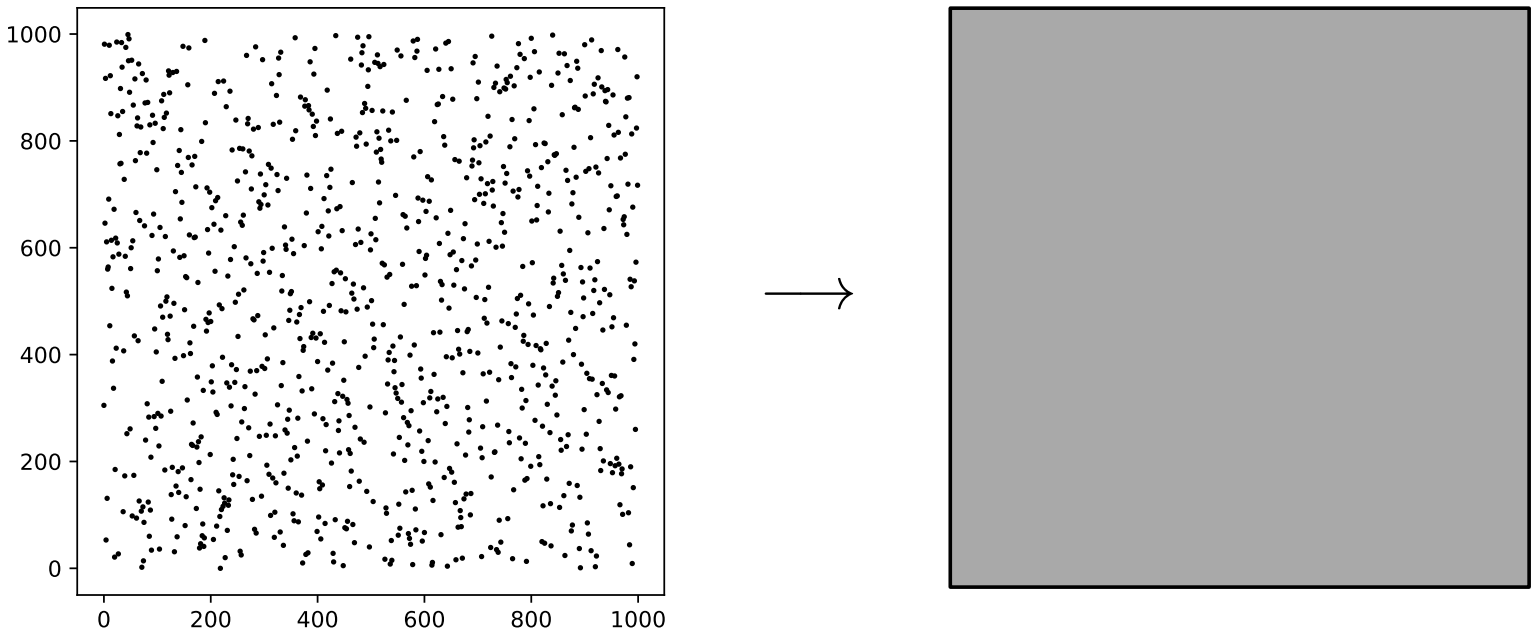

Uniform permutations converge a.s. to $\mathrm{Leb}_{[0,1]^2}$.

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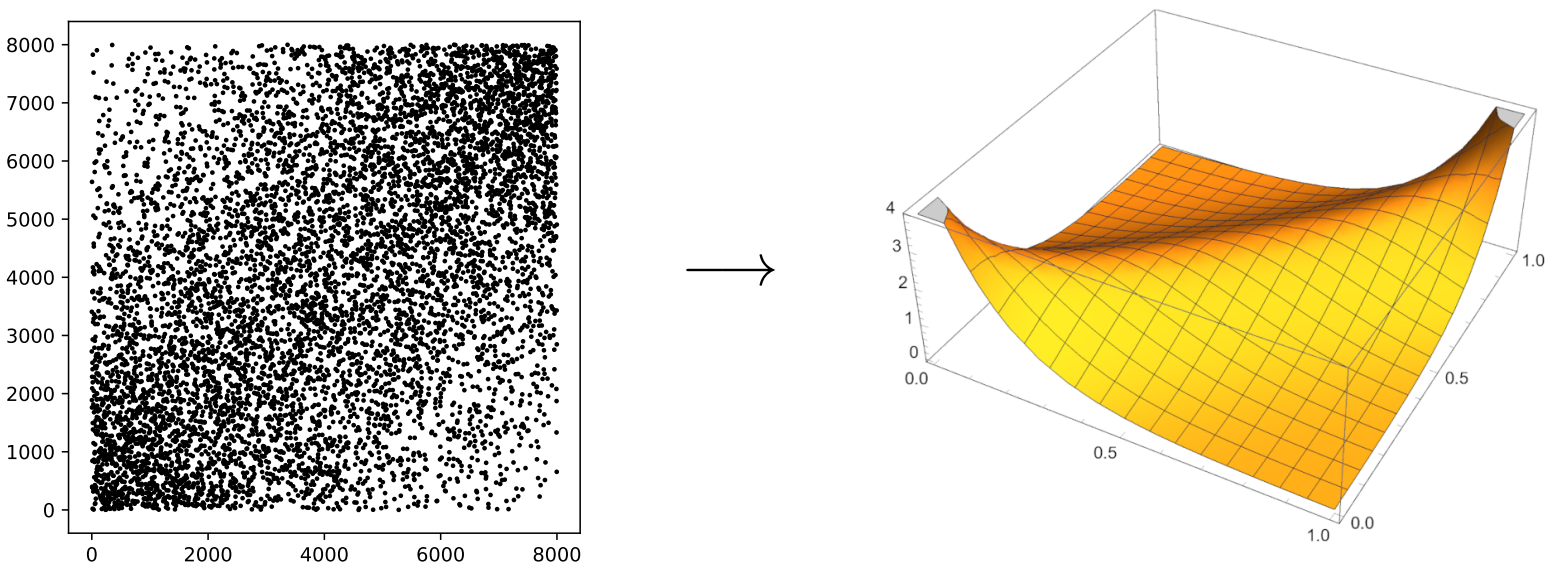

Mallows permutations with parameter $q_n = 1 - \frac{\beta}{n}$ converge in probability to the Mallows permuton μ_β.

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Hoppen et al '13

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Intuition

If (σ_n) converges towards μ then it should asymptotically behave, under suitable regularity assumptions, similarly to (σ_μ^n).

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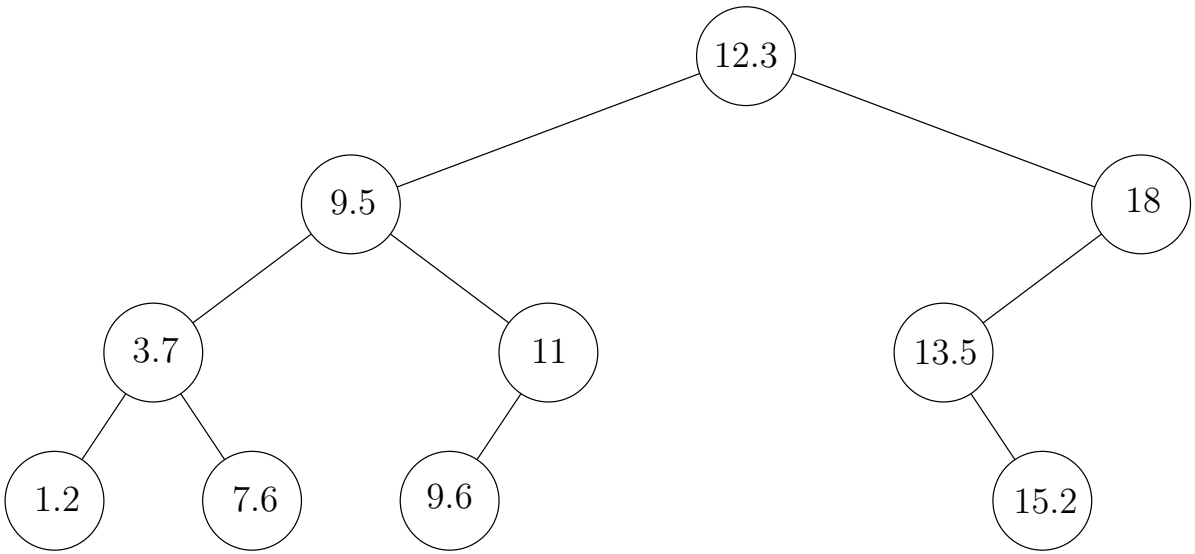
Binary Search Trees

A Binary Search Tree (BST) is a rooted labeled binary tree such that: for each vertex v, all labels of vertices in its left-subtree are smaller than that of v (resp. right-subtree, greater).



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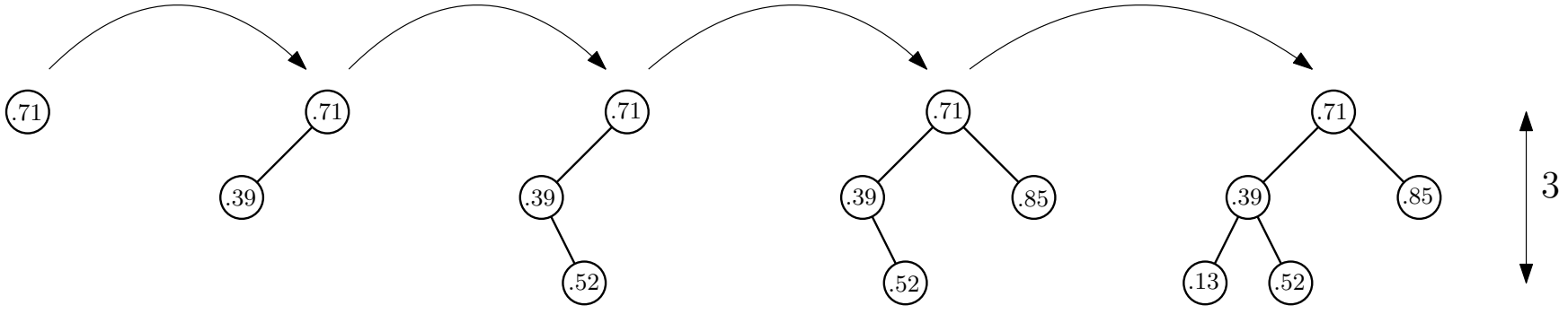
BST associated with a sequence of distinct numbers

Let $y = (y_1, \ldots, y_n)$ be a sequence of distinct numbers. We can construct a BST $\mathcal{T}\langle y \rangle$ by successively adding leaves with those labels.

Construction of $\mathcal{T}\langle y \rangle$ with $y = (.71, .39, .52, .85, .13)$:



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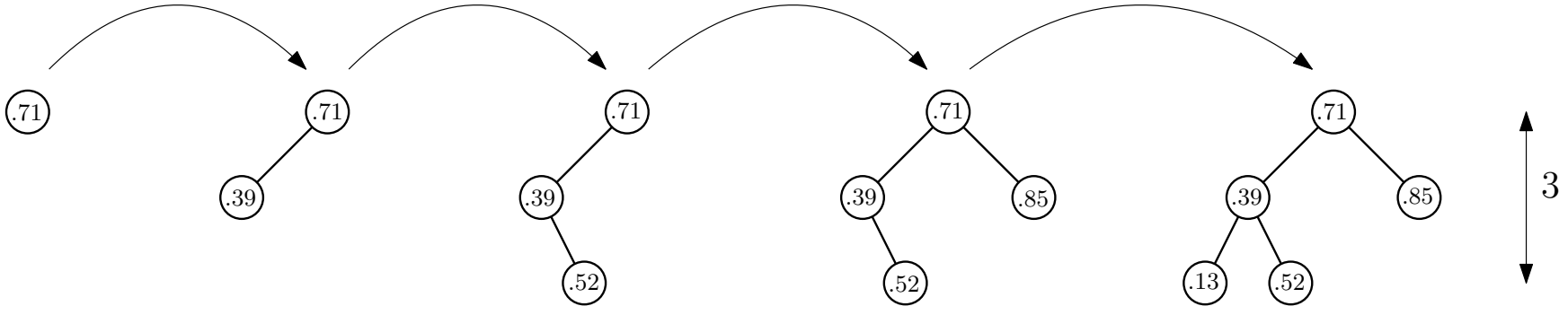

Rem: same shape as $\mathcal{T}\langle \sigma \rangle$ with $\sigma = (4, 2, 3, 5, 1)$.

II.b. Binary search trees

BST of a point process

Let $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{R}^2$ with no common x- or y-coordinate.
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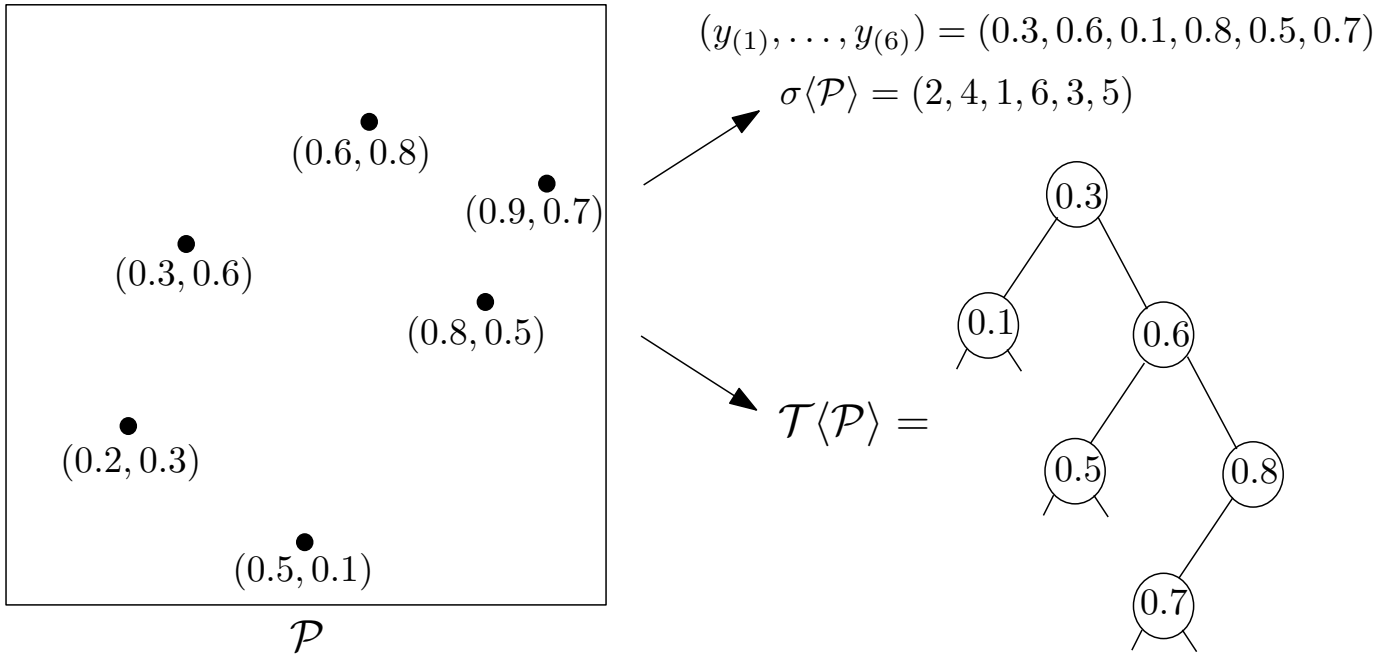

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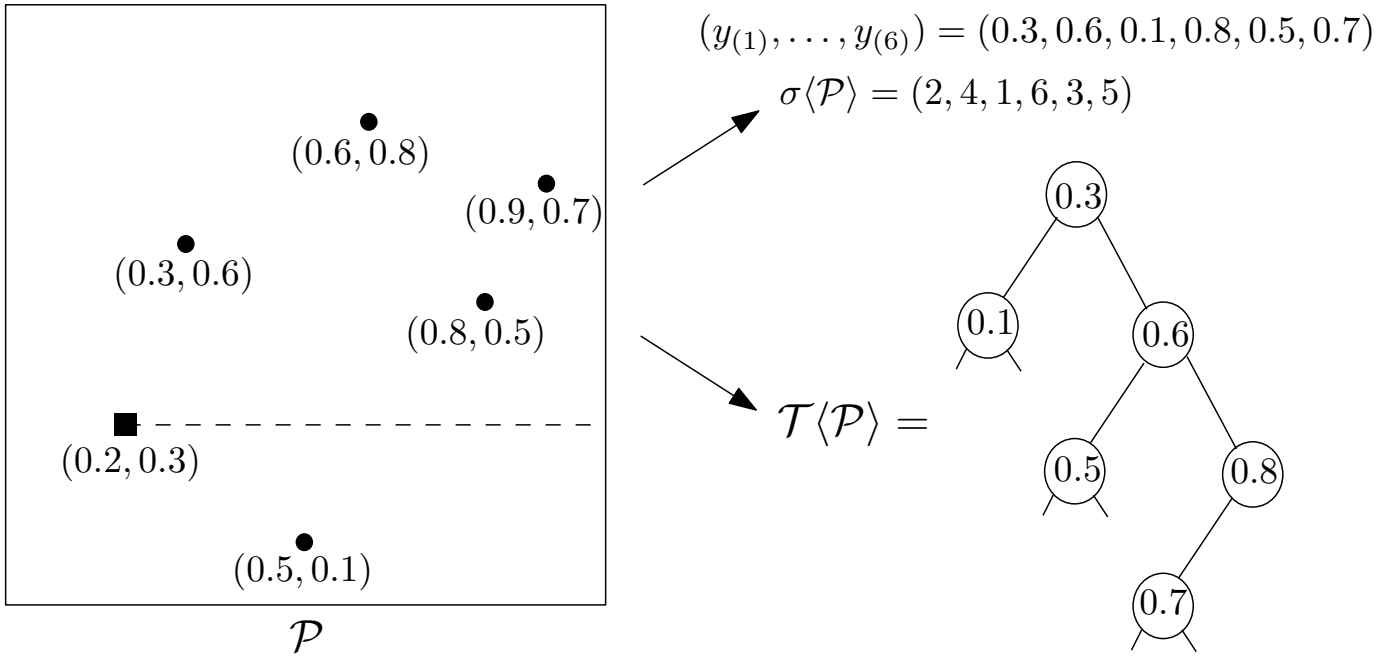

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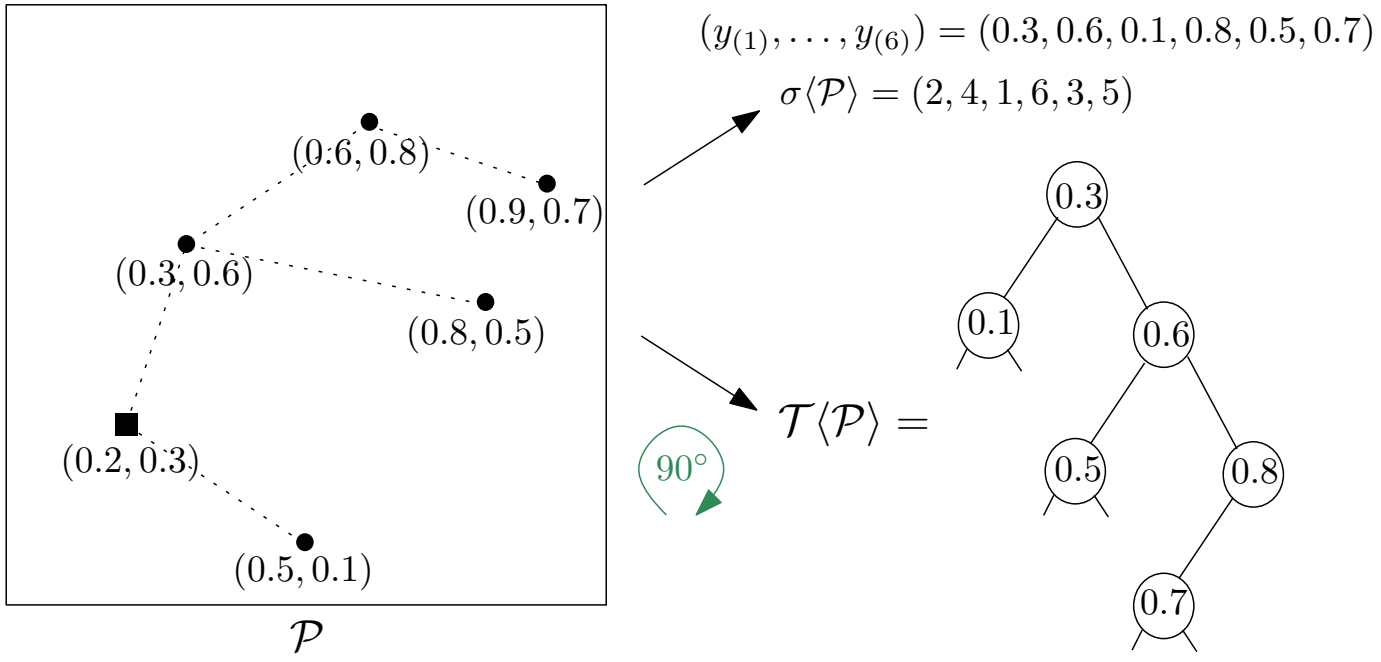

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III. BSTs of permuton samples

The height of a BST determines the complexity of operations such as lookup, addition or removal of data.

Q: Let σ be a (random) permutation. What is the height of $\mathcal{T}\langle\sigma\rangle$?

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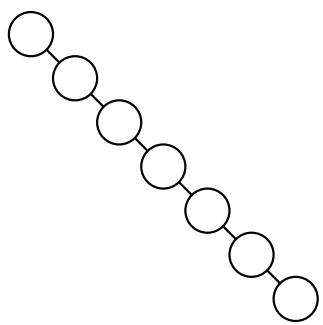
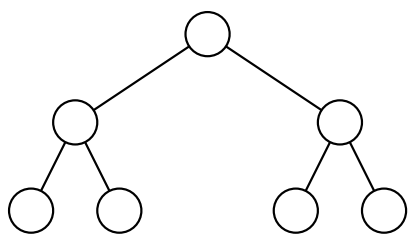
The height of a BST determines the complexity of operations such as lookup, addition or removal of data.

Q: Let σ be a (random) permutation. What is the height of $\mathcal{T}\langle\sigma\rangle$?

In general:

$$n \geq h\left(\mathcal{T}\langle\sigma(1), \ldots, \sigma(n)\rangle\right) \geq \log_2(n+1)$$





Depending on the law of σ, asymptotic results for $h\left(\mathcal{T}\langle\sigma\rangle\right)$?

III. BSTs of permuton samples

Devroye '86

If $\sigma^n \sim \mathrm{Unif}\left(\mathfrak{S}_n\right)$ then, as $n \to \infty$:

$$\frac{h\left(\mathcal{T}\langle\sigma^n\rangle\right)}{c^* \log n} \longrightarrow 1$$

in probability and in L^p for all $p \geq 1$, where $c^* \geq 2$ solves $c \log(2e/c) = 1$.

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Let $q \in [0,1]$. The Mallows distribution $\mathcal{M}_{n,q}$ is defined by:

$$\forall \sigma \in \mathfrak{S}_n, \quad \mathcal{M}_{n,q}(\sigma) \propto q^{\mathrm{inv}(\sigma)}\,.$$

Addario-Berry – Corsini '21

If $\sigma^n \sim \mathcal{M}_{n,q_n}$ where $n(1-q_n)/\log n \to 0$, then:

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III. BSTs of permuton samples

Assumption (A1):

μ has a bounded density ρ on $[0,1]^2$, which is continuous and positive on a neighborhood of $\{0\} \times [0,1]$.

Corsini – D. – Féray '24

Let μ be a permuton satisfying (A1). Then:

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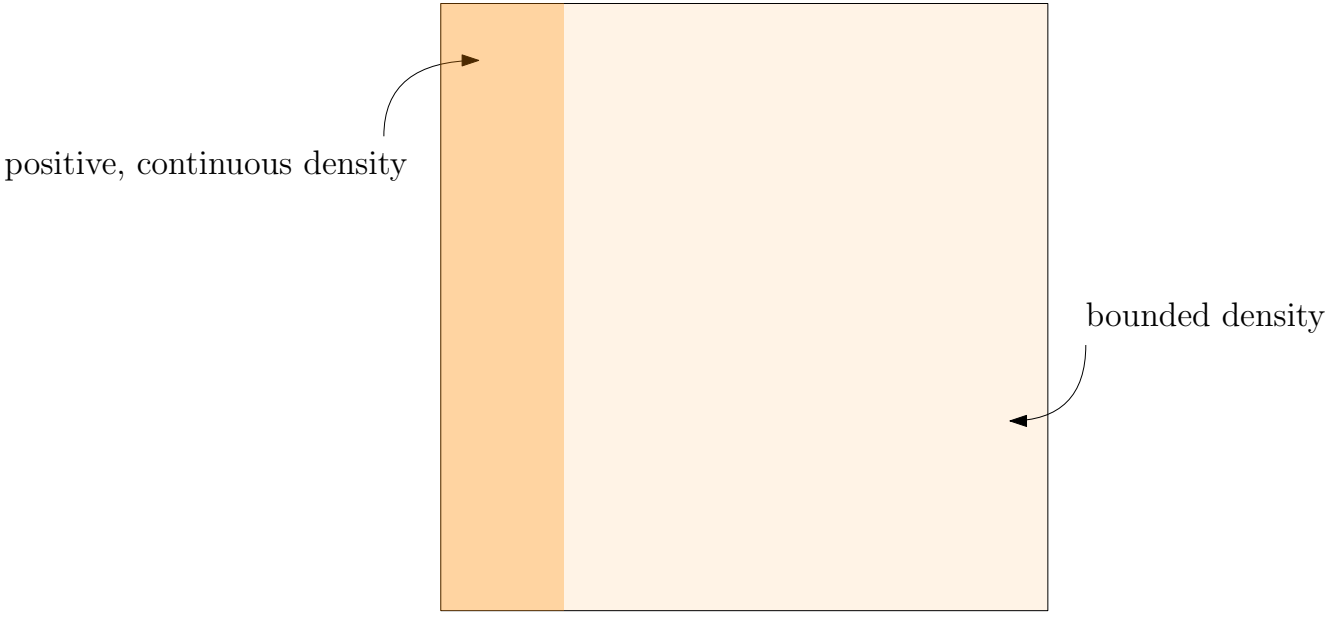

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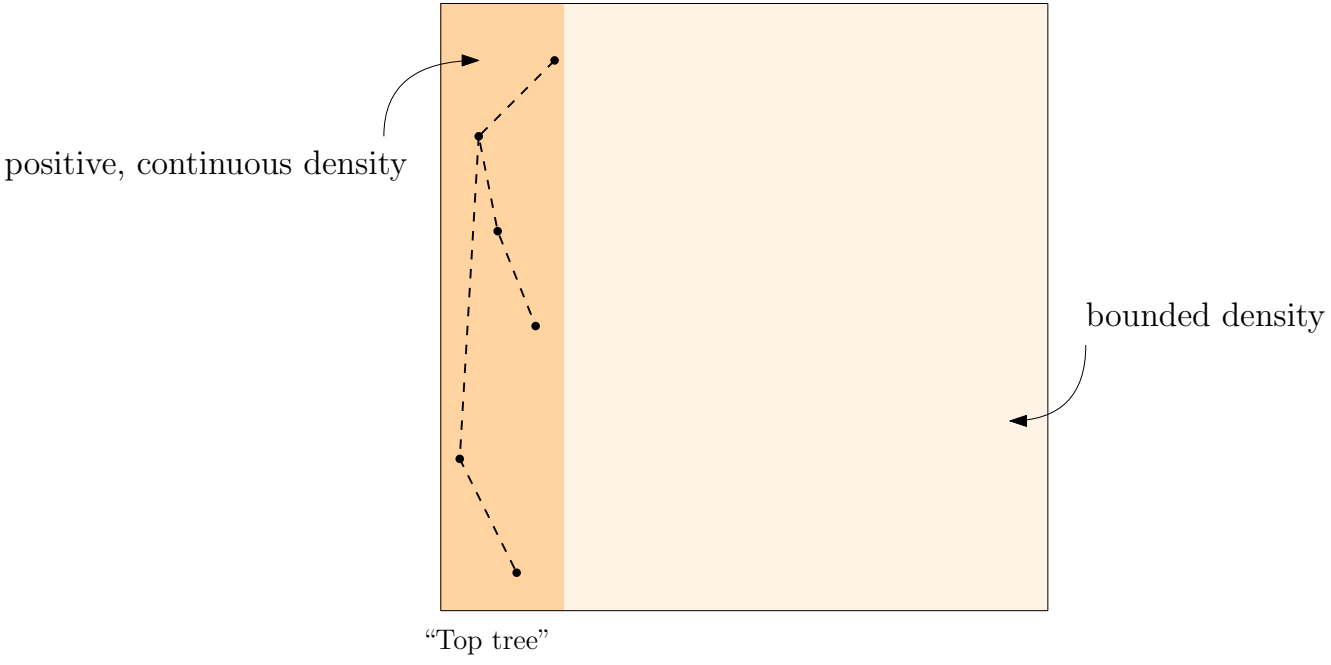

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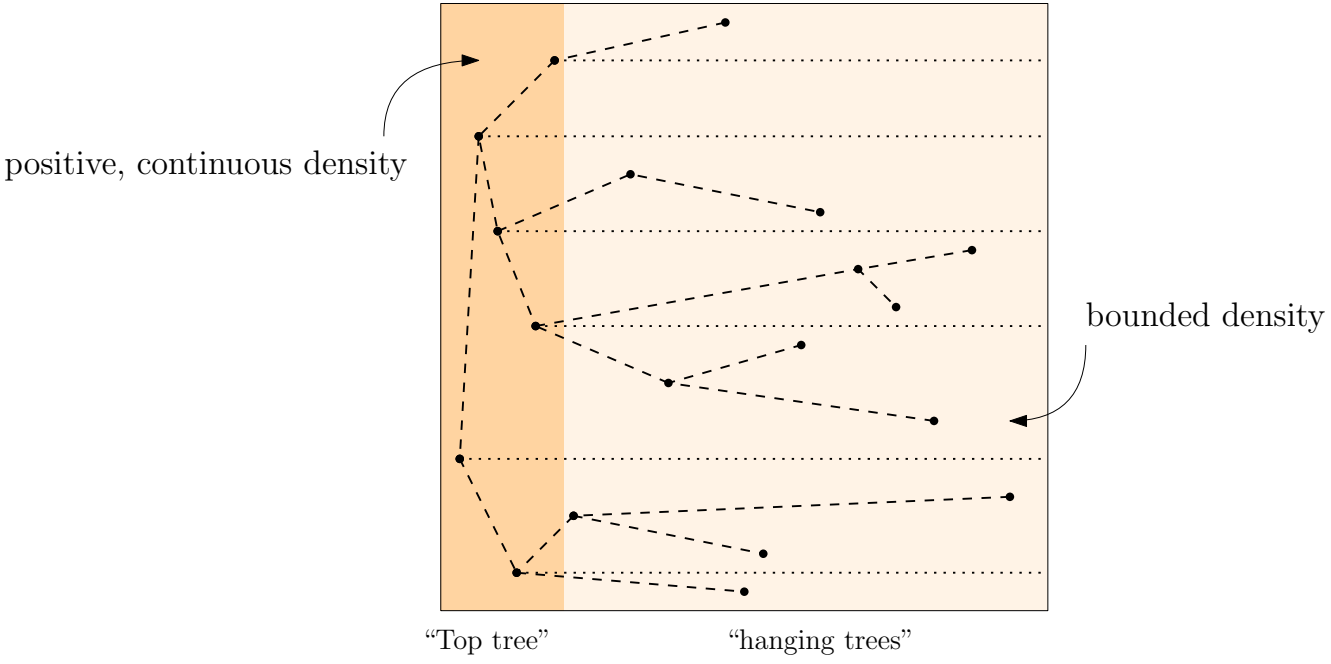

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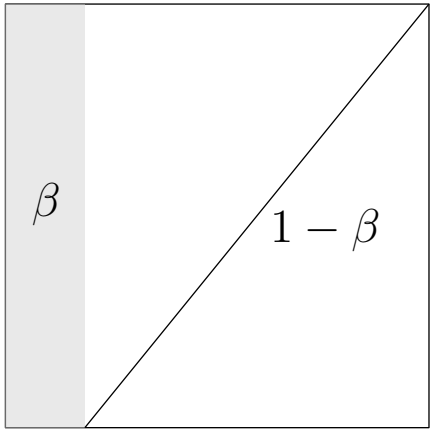
Corsini – D. – Féray '24

Let μ be a permuton satisfying (A1). Then:

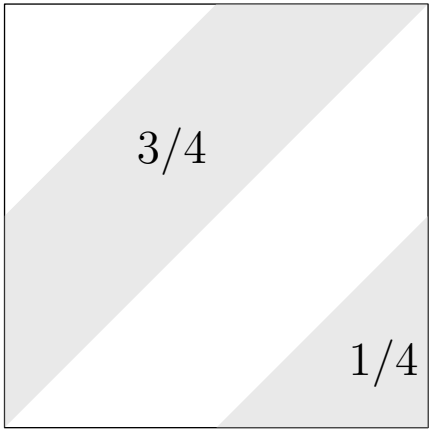
$$\frac{h\left(\mathcal{T}\langle\sigma_\mu^n\rangle\right)}{c^* \log n} \longrightarrow 1$$

in probability and in L^p for all $p \geq 1$.

Necessity of assumptions:

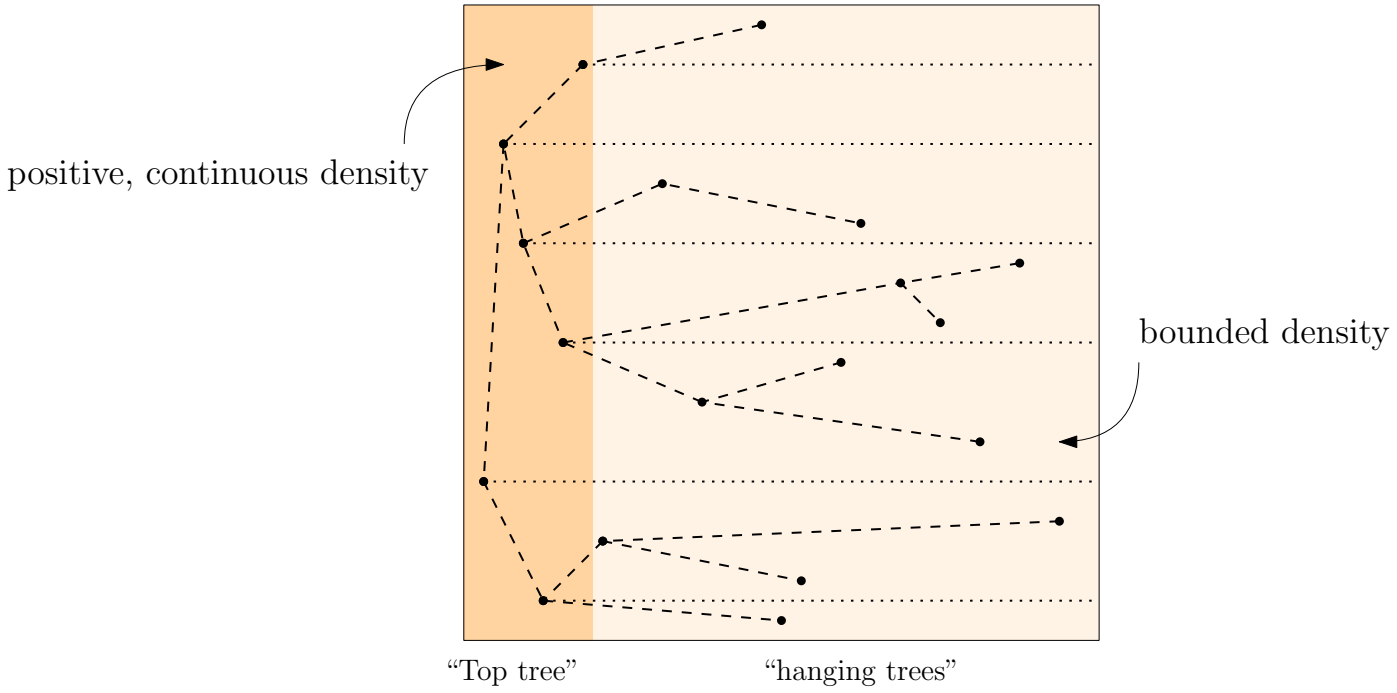


Height $\sim c_\beta \log n$ for some $c_\beta > c^*$



Height $\geq c\sqrt{n}$

III. BSTs of permuton samples



$$h\left(\mathcal{T}_{\text{top}}\right) \leq h\left(\mathcal{T}\langle\sigma_\mu^n\rangle\right) \leq h\left(\mathcal{T}_{\text{top}}\right) + 1 + \max_k h\left(\mathcal{T}_{\text{hanging}}^k\right)$$

Two steps:

1. $h\left(\mathcal{T}_{\text{top}}\right) \sim c^* \log n$;
2. $\max_k h\left(\mathcal{T}_{\text{hanging}}^k\right) = o\left(\log n\right)$.

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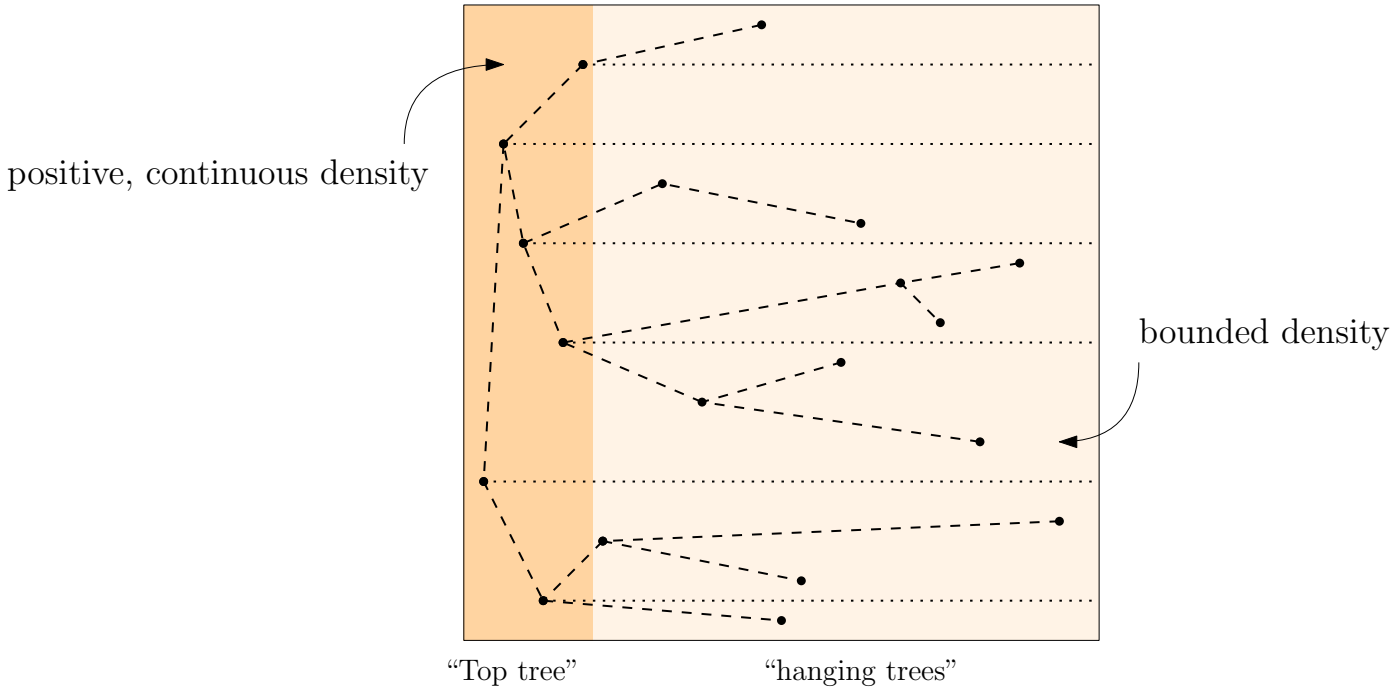

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Main technique:

local coupling with uniform permutations.

I. Models of non-uniform random permutations

II. A geometric construction

III. BSTs of permuton samples

IV. Pattern counts of conjugation-invariant permutations

V. Conclusion

IV. Conjugation-invariance

Definition

A random permutation τ is *conjugation-invariant* if for any given permutation π, it holds that $\pi \circ \tau \circ \pi^{-1} \overset{\text{law}}{=} \tau$.

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If τ is a permutation of $[n] := \{1, \ldots, n\}$, we write $t = (t_1, \ldots, t_n)$ for its *cycle type*, where t_p denotes the number of p-cycles in τ.

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Property

τ is conjugation-invariant

$$\Longleftrightarrow$$

conditionally given t, τ is a uniformly random t-cyclic permutation.

$\longrightarrow$ we may study uniform permutations with given cycle types.

IV. Conjugation-invariance

A few examples:

- Uniform permutations

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- Uniform involutions
- Uniform involutions with no fixed points

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and their generalizations.

IV. Conjugation-invariance

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- Uniform involutions with no fixed points
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and their generalizations.

Hamaker–Rhoades '22, D. '24

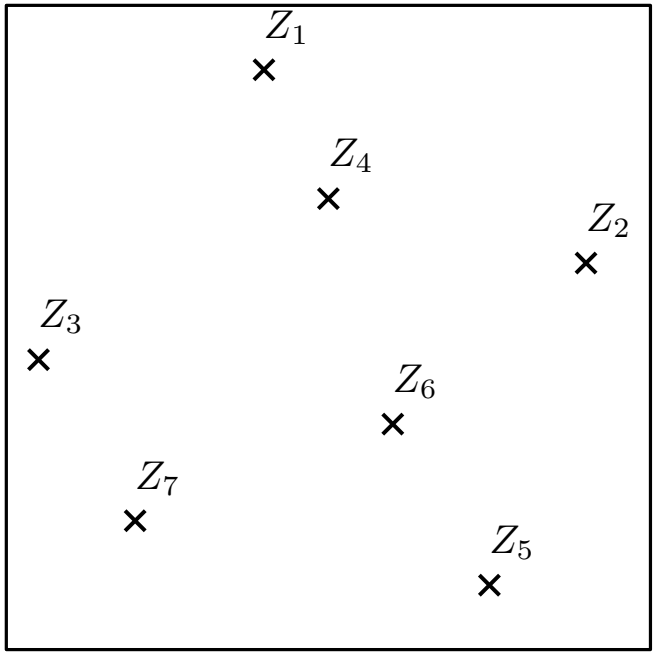
These permutations converge to $\alpha \mathrm{Leb}_{[0,1]^2} + (1-\alpha)\mathrm{Leb}_{\Delta}$ where Δ is the diagonal of $[0,1]^2$ and $1-\alpha$ is the asymptotic proportion of fixed points.

IV. Conjugation-invariance

Mapping a point set to a permutation

Let $\mathcal{P} = \{Z_1, \ldots, Z_n\} \subset [0,1]^2$ with distinct x-coordinates and distinct y-coordinates. Define $\tau = \mathrm{Perm}(\mathcal{P})$ by:

$\tau(i) = j$ iff the i-th point from the left is the j-th point from the bottom.



$\mathrm{Perm}(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7) = 4\ 2\ 7\ 6\ 3\ 1\ 5$

IV. Conjugation-invariance

Reminder: recovering uniform permutations

Let $\mathcal{P} = \{Z_1, \ldots, Z_n\}$ be i.i.d. $\mathrm{Unif}\left([0,1]^2\right)$. Then $\tau = \mathrm{Perm}\left(\mathcal{P}\right)$ is a uniform permutation of $[n]$.

IV. Conjugation-invariance

Reminder: recovering uniform permutations

Let $\mathcal{P} = \{Z_1, \ldots, Z_n\}$ be i.i.d. $\mathrm{Unif}\left([0,1]^2\right)$. Then $\tau = \mathrm{Perm}\left(\mathcal{P}\right)$ is a uniform permutation of $[n]$.

Recovering random involutions (Baik and Rains '99)

Let $\mathcal{P}^*$ be the mirror of $\mathcal{P}$ with respect to the diagonal of $[0,1]^2$. Then $\tau = \mathrm{Perm}\left(\mathcal{P} \cup \mathcal{P}^*\right)$ is a uniform involution of $[2n]$ with no fixed point.

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Idea: the set $\mathcal{P} \cup \mathcal{P}^*$ is symmetric w.r.t. the diagonal, and has the same x-coordinates as y-coordinates.
If (X, Y) has relative position (i, j) in this set, then (Y, X) has relative position (j, i). Thus $\tau(i) = j$ and $\tau(j) = i$.

IV. Conjugation-invariance

Fix a cycle type t of size n. Fix any t-cyclic permutation $\mathfrak{s}$.

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Recovering uniform t-cyclic permutations (D. '24)

Let $(U_i)_{1\leq i\leq n}$ be i.i.d. $\mathrm{Unif}([0,1])$. For each i, set $Z_i := \left(U_i, U_{\mathfrak{s}(i)}\right)$. Then $\tau = \mathrm{Perm}\left(Z_i, 1 \leq i \leq n\right)$ is a uniform t-cyclic permutation.

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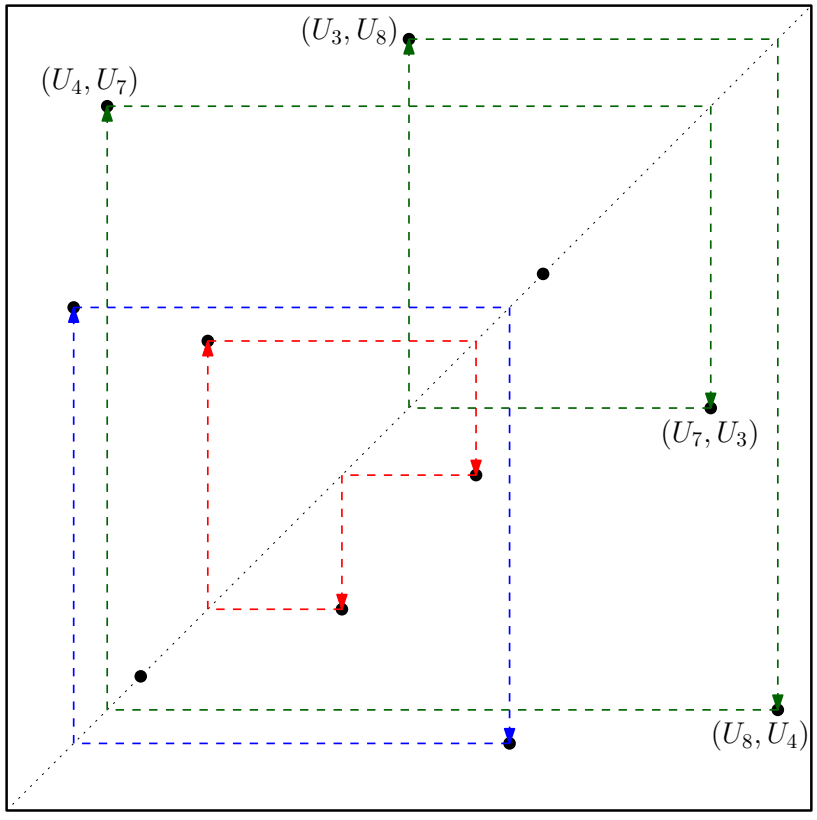

Example with:

$$\mathfrak{s} = (5)\circ(11)\circ(1,9)\circ(2,6,10)\circ(4,7,3,8)$$

The construction yields:

$$\tau = 8\ \ 10\ \ 3\ \ 7\ \ 4\ \ 11\ \ 5\ \ 1\ \ 9\ \ 6\ \ 2$$
$$= (3)\circ(9)\circ(1,8)\circ(4,7,5)\circ(2,10,6,11)$$

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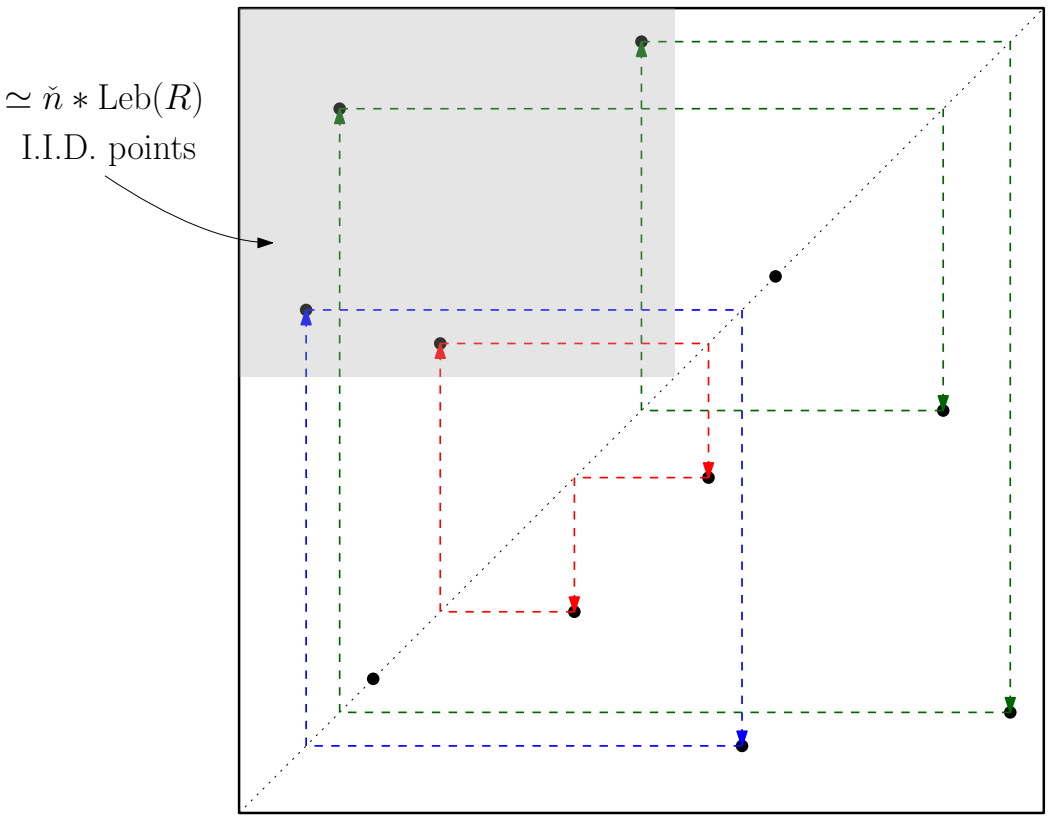

Property: if R is a rectangle which does not intersect the diagonal, it contains $\sim (n - t_1) \cdot \mathrm{Leb}(R)$ i.i.d. uniform points.

Corollary

Convergence towards $\alpha \mathrm{Leb}_{[0,1]^2} + (1-\alpha)\mathrm{Leb}_{\Delta}$.

IV. Conjugation-invariance

Pattern occurrences

Let $\tau = \tau(1)\ \tau(2)\ \dots\ \tau(n) \in \mathfrak{S}_n$. The *pattern* induced by τ on $I = (i_1 < \cdots < i_r)$ is the permutation $\mathrm{pat}\,(I, \tau) = \pi \in \mathfrak{S}_r$ defined by:

$$\forall j, k \in [r], \quad \pi(j) < \pi(k) \Longleftrightarrow \tau(i_j) < \tau(i_k).$$

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The number of (classical) occurrences of $\pi \in \mathfrak{S}_r$ in $\tau \in \mathfrak{S}_n$ is:

$$\mathrm{Occ}\,(\pi, \tau) := \sum_{I \in \binom{[n]}{r}} 1_{\mathrm{pat}(I,\tau)=\pi}$$

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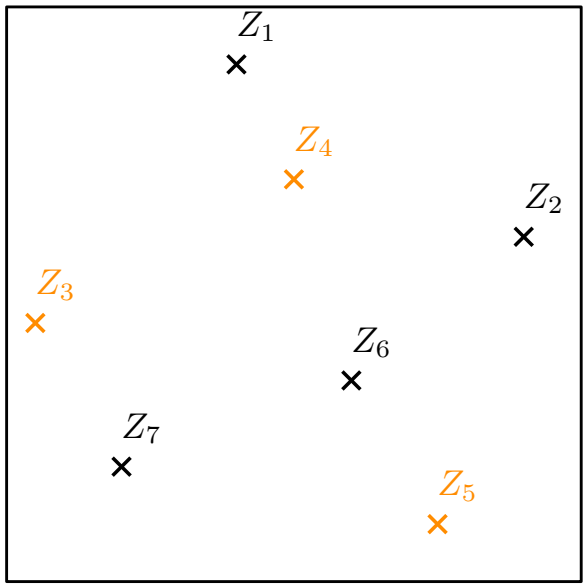
Ex: $\mathrm{Occ}\,(2\,1, \tau)$ is the number of inversions in τ.

IV. Conjugation-invariance

Context: $\tau = \mathrm{Perm}\,(Z_i, 1 \le i \le n)$.



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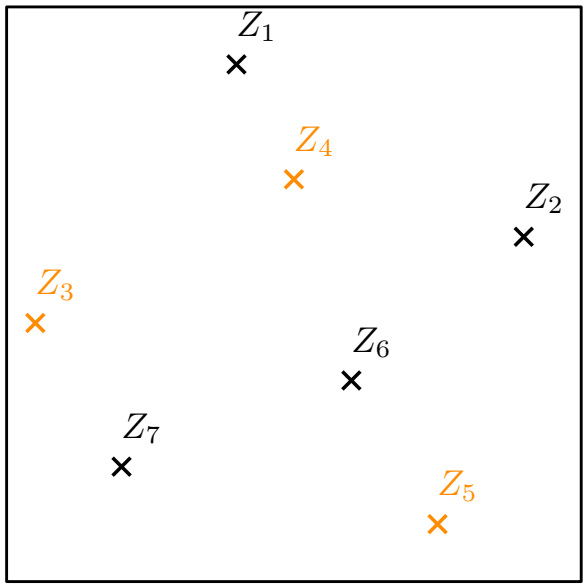

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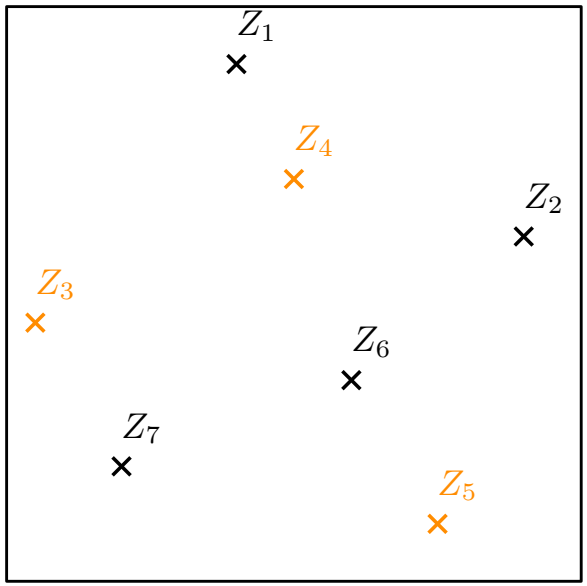

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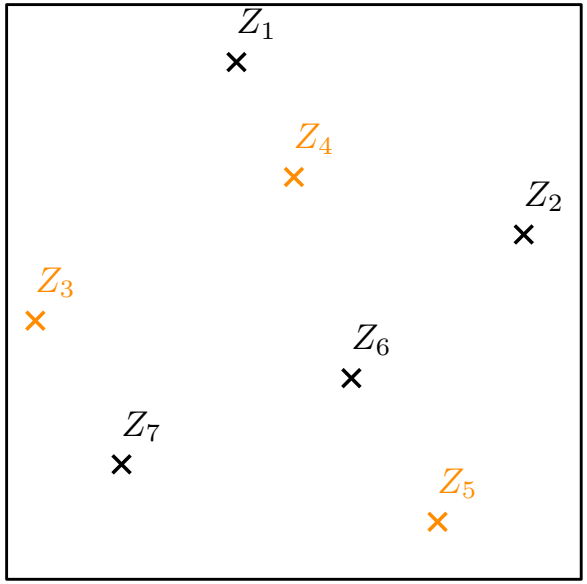

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The r.v.'s Z_i are not independent, but "almost"...

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Define a graph $\mathcal{L}_{\mathfrak{s}}$ with vertices $[n]$ and edges $i \sim \mathfrak{s}(i)$ for all $i \in [n]$.

Property: Dependency graph

If $I \subseteq [n]$ is an independent subset of $\mathcal{L}_{\mathfrak{s}}$, then the random variables $(Z_i)_{i \in I}$ are independent.

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$\mathcal{L}_{\mathfrak{s}}$ is a "nice" dependency graph: its degrees are bounded by 2. Most subsets ($\binom{n}{r} - \mathcal{O}(n^{r-1})$) are independent.

IV. Conjugation-invariance

Context: t is a cycle type of size n with $t_1 = 0$, $Z_i = \left(U_i, U_{\mathfrak{s}(i)}\right)$ for each $i \in [n]$, and $\tau = \mathrm{Perm}\left(Z_i, i \in [n]\right)$.

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\end{aligned}$$

where $\varphi(Z) := \mathbb{E}\left[1_{\mathrm{Perm}\left(Z, \hat{Z}_2, \ldots, \hat{Z}_r\right) = \pi} \;\middle|\; Z\right]$ and $\hat{Z}_i$ are i.i.d. $\mathrm{Unif}\left([0,1]^2\right)$.

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This is a sum of "almost" independent r.v.'s, for which a CLT is easily established.

IV. Conjugation-invariance

D. '24

If $t_1^{(n)} = o(\sqrt{n})$ and $2t_2^{(n)} = np_2 + o(n)$, then for any $\pi \in \mathfrak{S}_r$:

$$\frac{\mathrm{Occ}\left(\pi, \tau_n\right) - \binom{n}{r}\frac{1}{r!}}{n^{r-1/2}} \xrightarrow[n\to\infty]{} \mathcal{N}\left(0, \sigma^2\right)$$

where $\left((r-1)!\sigma\right)^2 = \mathrm{Var}\left[\varphi(U,V)\right] + p_2\mathrm{Cov}\left[\varphi(U,V), \varphi(V,U)\right]$.
Moreover $\sigma > 0$.

IV. Conjugation-invariance

D. '24

If $t_1^{(n)} = np_1 + o(\sqrt{n})$ and $2t_2^{(n)} = np_2 + o(n)$, then:

$$\left(\frac{\mathrm{Occ}_\pi(\tau_n) - \binom{n}{r}\mu_\pi^{p_1}}{n^{r-1/2}} \right)_{\pi \in \mathfrak{S}_r} \xrightarrow[n\to\infty]{} \mathcal{N}\left(0, \Sigma^{p_1,p_2}\right)$$

for some "explicit" $\mu_\pi^{p_1}$ and $\left(\Sigma_{\pi,\rho}^{p_1,p_2}\right)_{\pi,\rho\in\mathfrak{S}_r}$.

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$\to$ independently proved by Féray–Kammoun '23, using *weighted* dependency graphs

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D. '24

If $t_1^{(n)} = np_1 + o(\sqrt{n})$ and $2t_2^{(n)} = np_2 + o(n)$, then:

$$\left(\frac{\mathrm{Occ}_\pi(\tau_n) - \binom{n}{r}\mu_\pi^{p_1}}{n^{r-1/2}} \right)_{\pi \in \mathfrak{S}_r} \xrightarrow[n\to\infty]{} \mathcal{N}\left(0, \Sigma^{p_1,p_2}\right)$$

for some "explicit" $\mu_\pi^{p_1}$ and $\left(\Sigma_{\pi,\rho}^{p_1,p_2}\right)_{\pi,\rho\in\mathfrak{S}_r}$.

$\to$ independently proved by Féray–Kammoun '23, using *weighted* dependency graphs

$\to$ our approach also yields explicit bounds on the speed of convergence, deviation inequalities...

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... and applies to many other statistics!

V. Conclusion

The geometric approach is well suited to prove universality results for "visual" statistics, such as:

- Longest monotone subsequences;
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Thank you for your attention!

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