Asymptotics of Longest Increasing Subsequences in random permutations

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Context

LIS and RS-shape of a permutation

Let σ be a permutation of $\{1,\dots,n\}$. An **increasing subsequence** of σ is a sequence of indices $i_1 < \cdots < i_\ell$ satisfying $\sigma(i_1) < \cdots < \sigma(i_\ell)$. Denote by $\mathrm{LIS}_k(\sigma)$ the maximal size of a union of k increasing subsequences of σ.

Robinson-Schensted's correspondence maps σ to a pair of standard Young tableaux with a common shape. For example the RS tableaux of the permutation $\sigma = 4\,2\,7\,6\,1\,3\,5$ are:

$P(\sigma) =$

1	3	5
2	6	
4	7	

, $Q(\sigma) =$

1	3	7
2	4	
5	6	

.

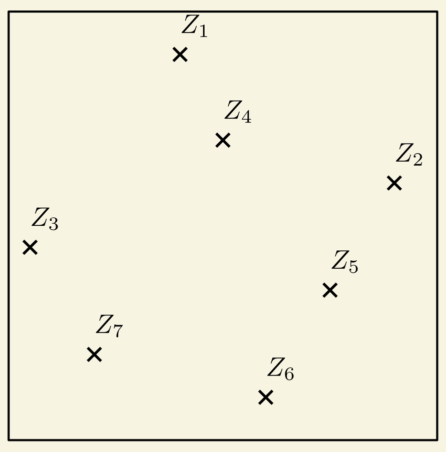
The link between these objects: **Greene's theorem**, stating that $\mathrm{LIS}_k(\sigma)$ is the number of boxes in the k first rows of σ's RS tableaux.

Random permutations sampled from a permuton

Define a **permuton** to be a probability measure on $[0,1]^2$ with uniform marginals.

If μ is a permuton, use it to sample n i.i.d. points $Z_1,\dots,Z_n$. Then set $\sigma(i)=j$ iff the i-th point from the left is j-th from the bottom.

$\rightsquigarrow$ random permutation of law $\mathrm{Sample}_n(\mu)$.



$\mathrm{Perm}(Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7) = 4\,2\,7\,6\,1\,3\,5$

Problem

What are the asymptotics as $n\to\infty$ of $\mathrm{LIS}_k(\sigma_n)$ and $\mathrm{RS}(\sigma_n)$ if $\sigma_n \sim \mathrm{Sample}_n(\mu)$?

Current litterature: $\sqrt{n}$ scaling limits

The well known uniform case

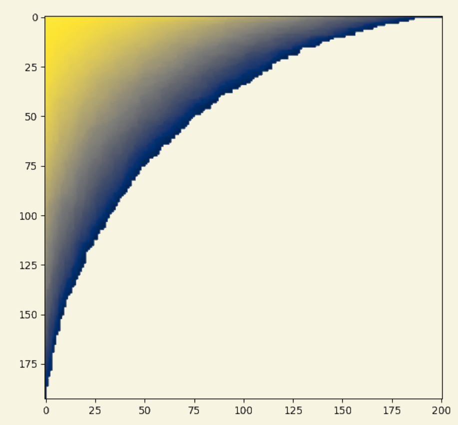
When $\mu = \mathrm{Leb}$ on $[0,1]^2$, σ_n is uniform. Vershik and Kerov [5] showed that

$$\frac{\mathrm{LIS}_1(\sigma_n)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 2$$

in probability.

More generally: $\sqrt{n}$ scaling limit for the RS shape of σ_n, *i.e.* non-trivial limit of $\frac{1}{n}\mathrm{LIS}_{x\sqrt{n}}(\sigma_n)$ for $x\in[0,2]$.

Below, the P-tableau of a size 10000 uniform permutation:



When the permuton has a density

When μ has a positive $\mathcal{C}^1_b$ density ρ on $[0,1]^2$, Deuschel and Zeitouni [1] proved:

$$\frac{\mathrm{LIS}_1(\sigma_n)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} K_\rho$$

in probability, for some positive constant K_ρ defined by a variational problem.

Sjöstrand [4] also investigated the limit shape of σ_n after $\sqrt{n}$ renormalization, generalizing the limit shape of uniform permutations.

When the permuton density is allowed to diverge, we showed in [3] that $\mathrm{LIS}(\sigma_n)$ could behave as **any given power of n** (up to logarithmic factors).

Our study: linear scaling limits

Shape of a permuton

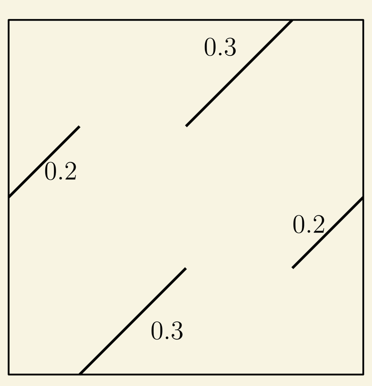
Say a subset of $[0,1]^2$ is **nondecreasing** when it is totally ordered for the natural partial order of the plane. Define

$$\widetilde{\mathrm{LIS}}_k(\mu) := \max_{A_1,\dots,A_k \text{ all nondecreasing}} \mu(A_1\cup\cdots\cup A_k).$$

This extends in a sense the notion of longest increasing subsequence to permutons. We can then define the **RS shape of a permuton** μ as:

$$\widetilde{\mathrm{sh}}(\mu) := \left(\widetilde{\mathrm{LIS}}_k(\mu) - \widetilde{\mathrm{LIS}}_{k-1}(\mu)\right)_{k\in\mathbb{N}^*}.$$

Below, a permuton with finite RS shape $(0.6, 0.4)$:



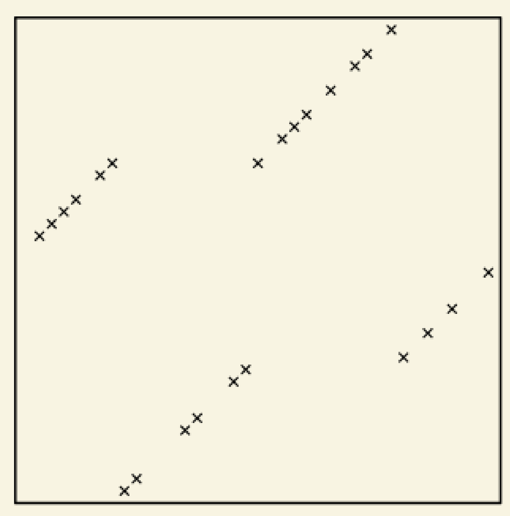
Convergence results

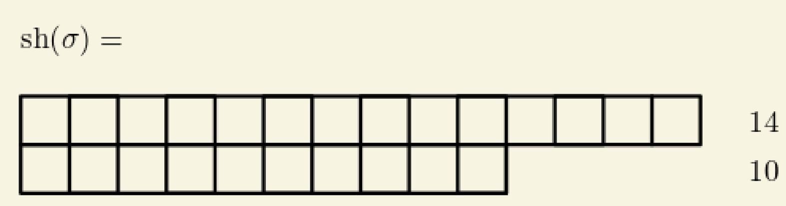
Theorem [2]. If $\sigma_n \sim \mathrm{Sample}_n(\mu)$ then for any $k\in\mathbb{N}^*$ the following convergence holds almost surely:

$$\frac{\mathrm{LIS}_k(\sigma_n)}{n} \underset{n\to\infty}{\longrightarrow} \widetilde{\mathrm{LIS}}_k(\mu).$$

In particular we deduce the almost sure **convergence of RS shapes**. Also : partial large deviation results [2].

Below, a permutation of size 24 sampled from the previous permuton.

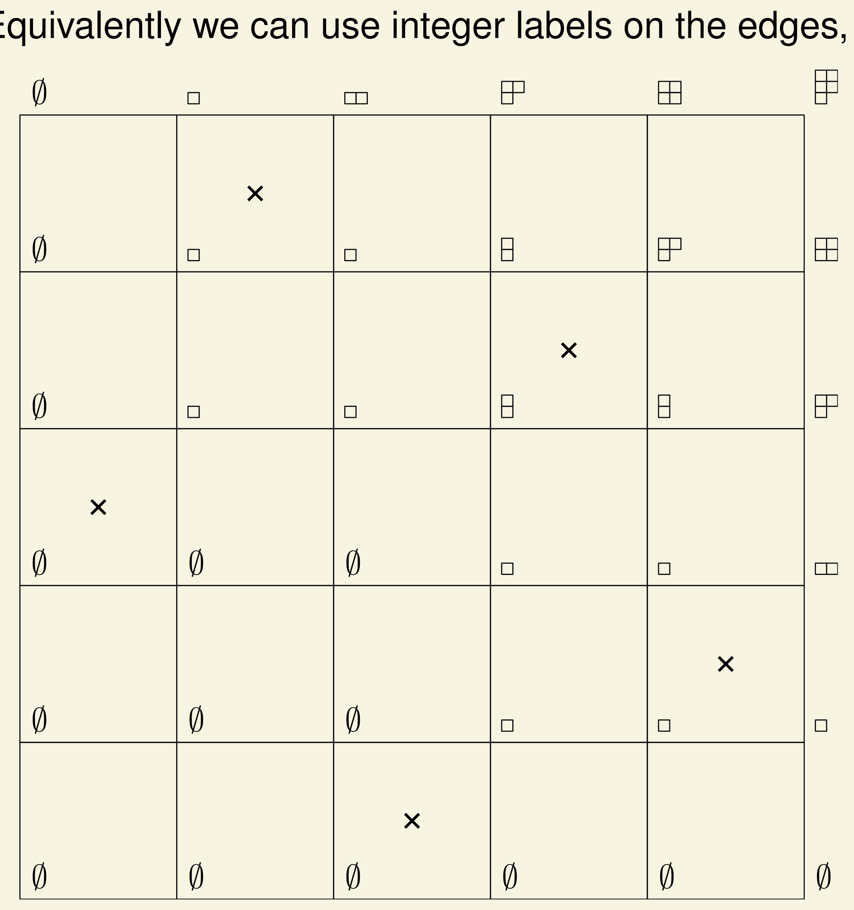


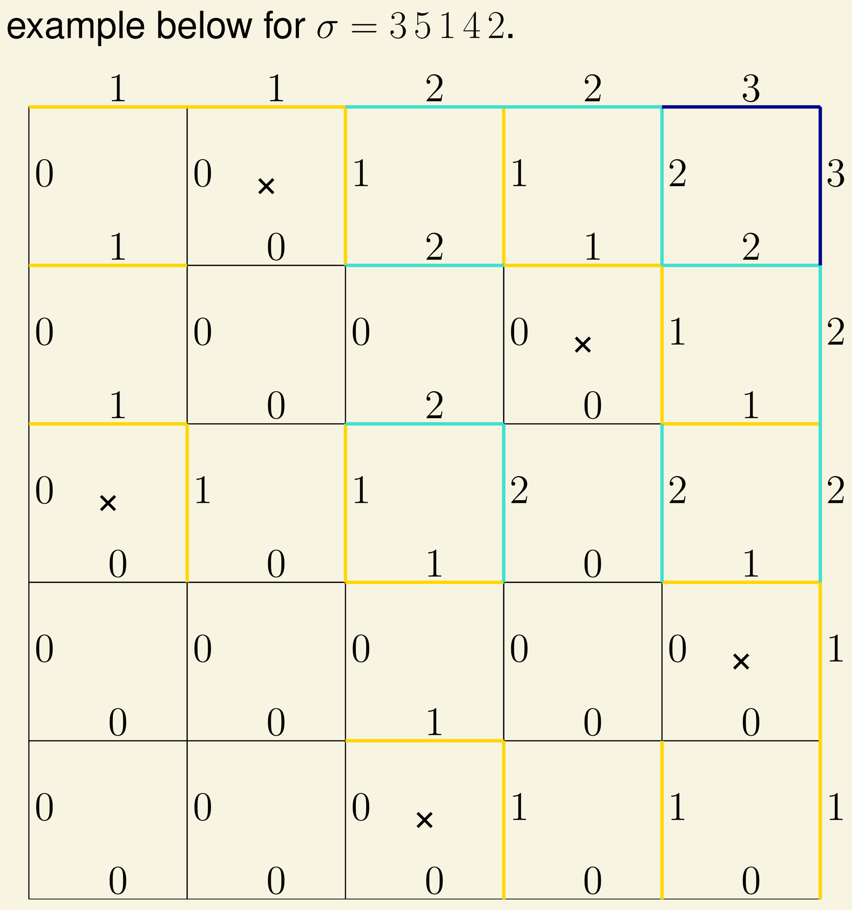


A discrete algorithm: Fomin inverse rules

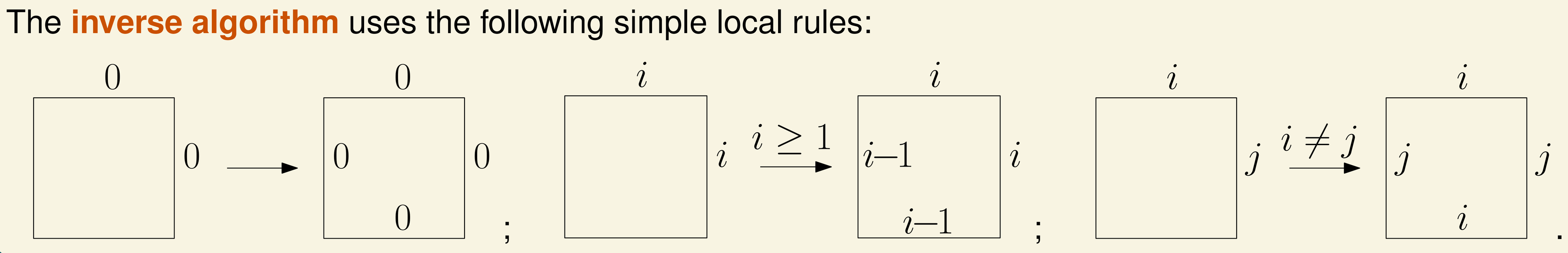
Robinson-Schensted's correspondence can be computed through **Fomin's local rules**: label each vertex of a grid with a diagram, and use local rules to construct these diagrams step by step. On the right and top borders: P- and Q-tableaux of the permutation.

Equivalently we can use integer labels on the edges, see example below for $\sigma = 3\,5\,1\,4\,2$.





The **inverse algorithm** uses the following simple local rules:



A continuous analogue: differential equations

Define the **RS tableaux of a permuton** μ as

$$\widetilde{\mathrm{RS}}(\mu) := \left(\widetilde{\lambda}^\mu(1,\cdot), \widetilde{\lambda}^\mu(\cdot,1)\right)$$

where for any $(x,y)\in[0,1]^2$:

$$\widetilde{\lambda}^\mu(x,y) = \left(\widetilde{\lambda}^\mu_k(x,y)\right)_{k\in\mathbb{N}^*} := \widetilde{\mathrm{sh}}\left(\mu|_{[0,x]\times[0,y]}\right)$$

Previous results imply **convergence of RS tableaux**. Moreover Fomin's inverse local rules yield differential equations in the permuton limit:

Theorem [2]. Suppose $\widetilde{\mathrm{LIS}}_r(\mu)=1$ for some $r\in\mathbb{N}^*$ and let $(x,y)\in]0,1]^2$ be s.t.

$$\alpha_k := \partial_x^- \widetilde{\lambda}_k(x,y) \quad\text{and}\quad \beta_k := \partial_y^- \widetilde{\lambda}_k(x,y)$$

for $k\in[1,r]$ all exist. Then for any $s,t\geq 0$ and $k\in[1,r]$:

$$\lim_{\epsilon\to 0^+} \frac{\widetilde{\lambda}_k(x,y)-\widetilde{\lambda}_k(x-t\epsilon, y-s\epsilon)}{\epsilon} = \phi\Big((t\alpha_i)_{k\leq i\leq r}, (s\beta_i)_{k\leq i\leq r}\Big)$$

for some continuous function ϕ.

Informally: asymptotically and locally, **σ_n's random edge labels behave as if they were ordered in a decreasing way**.

In the proof we introduce an adapted equivalence relation on words of labels and show that it is implied by **Knuth equivalence**. Global idea: the P-tableau of a long random word on bounded letters is similar to the P-tableau of this word's decreasing reordering.

References

[1] Jean-Dominique Deuschel and Ofer Zeitouni. "Limiting curves for i.i.d. records". In: *The Annals of Probability* 23.2 (1995), pp. 852–878.

[2] Victor Dubach. "Increasing subsequences of linear size in random permutations and the Robinson-Schensted tableaux of permutons". In preparation. 2023.

[3] Victor Dubach. "Locally uniform random permutations with large increasing subsequences". Submitted, preprint arXiv:2301.07658. 2023.

[4] Jonas Sjöstrand. "Monotone subsequences in locally uniform random permutations". In: (2022). arXiv: 2207.11505.

[5] Anatoli M. Vershik and Sergueĭ V. Kerov. "Asymptotics of the Plancherel measure of the symmetric group and the limit form of Young tableaux". In: *Soviet Math. Dokl.* 18 (1977), pp. 527–531.