Large increasing subsequences in random permutations, and the Robinson-Schensted tableaux of permutons

Victor Dubach
under the supervision of Valentin Féray

Cortipom workshop, CIRM July 2023

Motivation

General question:
what are the asymptotic properties of a sequence of random permutations (σ_n) as $n \to \infty$?

Possible models of random permutations: uniform, uniform in some restricted class, biased by some statistic (Mallows, Ewens)...

Possible properties to study: cycle structure, patterns, records, descents...

Motivation

General question:
what are the asymptotic properties of a sequence of random permutations (σ_n) as $n \to \infty$?

Possible models of random permutations: uniform, uniform in some restricted class, biased by some statistic (Mallows, Ewens)...

Here $\to$ randomly sampled with a geometric point of view. Related to a scaling limit: permutons.

Possible properties to study: cycle structure, patterns, records, descents...

Motivation

General question:
what are the asymptotic properties of a sequence of random permutations (σ_n) as $n \to \infty$?

Possible models of random permutations: uniform, uniform in some restricted class, biased by some statistic (Mallows, Ewens)...

Here $\to$ randomly sampled with a geometric point of view. Related to a scaling limit: permutons.

Possible properties to study: cycle structure, patterns, records, descents...

Here $\to$ longest increasing subsequences (LIS) and Robinson-Schensted (RS) correspondence.

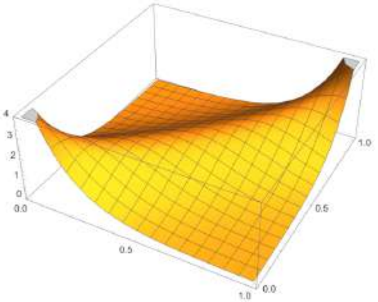
Outline

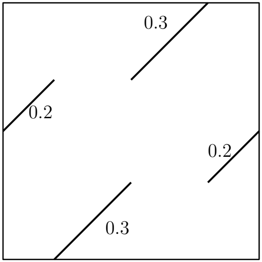
- Permuton theory and links with permutations
- Asymptotics of LIS and RS for sampled random permutations under density
- The RS shape of permutons, linear asymptotics
- The RS tableaux of permutons, linear asymptotics
- Fomin's inverse algorithm and its continuous analog

Permuton theory

The space of permutons

A *permuton* is a probability measure on $[0,1]^2$ whose marginals are both uniform. With weak convergence topology, the set of permutons form a metric compact space.





From permutations to permutons

Embedding permutations into permutons

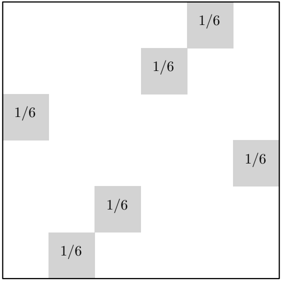
To each permutation $\sigma \in \mathfrak{S}_n$ we can associate a permuton μ_σ with density

$$n \sum_{i=1}^{n} 1_{\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{\sigma(i)-1}{n}, \frac{\sigma(i)}{n}\right]}$$

with respect to the Lebesgue measure on $[0,1]^2$.

$\sigma = 4\ 1\ 2\ 5\ 6\ 3$ $\longrightarrow$

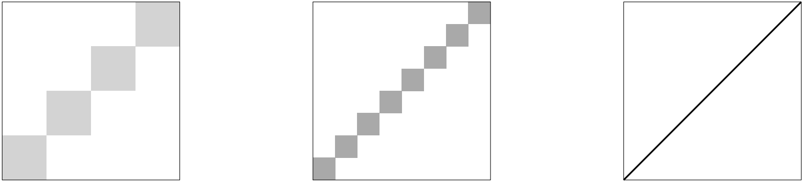
(one-line notation $\sigma = \sigma(1) \ldots \sigma(n)$)



From permutations to permutons

Convergence of a sequence of permutations

If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of permutations and μ is a permuton, we say that $(\sigma_n)_{n\in\mathbb{N}}$ converges towards μ when $\mu_{\sigma_n} \underset{n\to\infty}{\longrightarrow} \mu$ weakly.



From permutations to permutons

Pattern densities

Let $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$ with $k \leq n$. We say that a sequence of indices $i_1 < \cdots < i_k$ induces τ in σ if $\sigma(i_1), \ldots, \sigma(i_k)$ have the same relative order as $\tau(1), \ldots, \tau(k)$.
Denote by $\mathrm{occ}(\tau, \sigma)$ the number of such sequences and by $\mathrm{dens}(\tau, \sigma) := \binom{n}{k}^{-1} \mathrm{occ}(\tau, \sigma)$ their proportion.

$\mathrm{occ}(2\ 1\ 3\ ,\ 3\ 1\ 4\ 2\ 5) = 3$

Theorem [Hoppen et al '13]

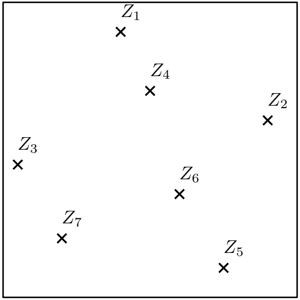
A sequence of permutations (σ_n) converges to a limit permuton iff for any permutation τ, the sequence $\big(\mathrm{dens}(\tau, \sigma_n)\big)$ converges.

Permutons are a "sampling limit theory" for permutations, just like graphons are for dense graphs...

From permutons to permutations

Sampling a random permutation from a permuton

Let $Z_1, \dots, Z_n \in [0,1]^2$ with no common x- or y-coordinate. Define a permutation σ such that $\sigma(i) = j$ whenever the i-th point from the left is the j-th point from the bottom.
If $Z_1, \dots, Z_n$ are distributed i.i.d. under a permuton μ, denote by $\mathrm{Sample}_n(\mu)$ the law of this random permutation.



$\mathrm{Perm}(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7) = 4\ 2\ 7\ 6\ 3\ 1\ 5$

From permutons to permutations

Theorem [Hoppen et al '13]

Let μ be a permuton and for each $n \in \mathbb{N}$, $\sigma_n \sim \mathrm{Sample}_n(\mu)$. Then (σ_n) almost surely converges towards μ:

$$\mu_{\sigma_n} \underset{n\to\infty}{\longrightarrow} \mu \quad \text{weakly a.s.}$$

As a consequence, permutations are dense in the space of permutons.

Our goal: asymptotic properties of $\boxed{\sigma_n \sim \mathrm{Sample}_n(\mu)}$.

More precisely: longest increasing subsequences, RS shape and tableaux.

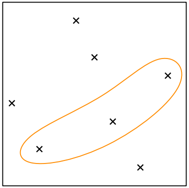
- Permuton theory and links with permutations
- Asymptotics of LIS and RS for sampled random permutations under density
- The RS shape of permutons, linear asymptotics
- The RS tableaux of permutons, linear asymptotics
- Fomin's inverse algorithm and its continuous analog

LIS and RS

Longest increasing subsequences

Let $\sigma \in \mathfrak{S}_n$. An increasing subsequence of σ is a sequence of indices $i_1 < \cdots < i_\ell$ such that $\sigma(i_1) < \cdots < \sigma(i_\ell)$. Denote by $\mathrm{LIS}(\sigma)$ the maximum size of an increasing subsequence of σ.

$$\mathrm{LIS}(4\ \underline{2}\ 7\ 6\ \underline{3}\ 1\ \underline{5}) = 3$$



⟵ up-right path of points

Longest k-increasing subsequences

For any $k \in \mathbb{N}$, denote by $\mathrm{LIS}_k(\sigma)$ the maximum size of a (disjoint) union of k increasing subsequences of σ.

$$\mathrm{LIS}_3(4\ 2\ 7\ 6\ 3\ 1\ 5) = 6$$

LIS and RS

Robinson-Schensted correspondence

The Robinson-Schensted correspondence is a bijection $\sigma \in \mathfrak{S}_n \mapsto \mathrm{RS}(\sigma) = \big(P(\sigma), Q(\sigma)\big)$ between permutations of size n and pairs of standard Young tableaux with the same shape $\mathrm{sh}(\sigma)$ on n boxes.
Note: this correspondence is also defined on words.

$\mathrm{RS}(4\ 2\ 7\ 6\ 3\ 1\ 5) =$

1	3	5
2	6	
4		
7		

,

1	3	7
2	4	
5		
6		

Greene's theorem

$\mathrm{LIS}_k(\sigma)$ is the number of boxes in the first k rows of $\mathrm{sh}(\sigma)$.

Asymptotics in the uniform case

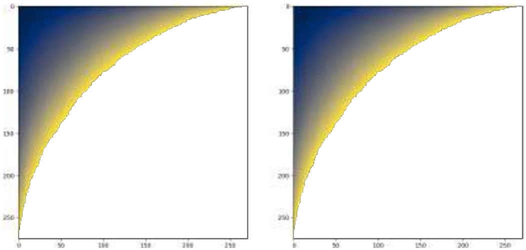
When $\mu = \mathrm{Leb}_{[0,1]^2}$, $\mathrm{Sample}_n(\mu)$ is the uniform law on $\mathfrak{S}_n$.

Solution to Ulam-Hammersley's problem [Vershik-Kerov '77]

If for each $n \in \mathbb{N}^*$, σ_n is a uniformly random permutation of size n:

$$\frac{1}{\sqrt{n}} \mathrm{LIS}(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} 2.$$

Also: $\sqrt{n}$ scaling limit for the shape (proved by Logan and Shepp simultaneously).



Asymptotics in the locally uniform case

Theorem [Deuschel-Zeitouni '95]

If μ has a **bounded** ($\mathcal{C}^1$, positive) density ρ then:

$$\frac{1}{\sqrt{n}}\mathrm{LIS}(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} \sup \int_0^1 2\sqrt{\rho\big(x(t),y(t)\big)x'(t)y'(t)}dt$$

where the supremum is taken over all increasing curves (x, y).

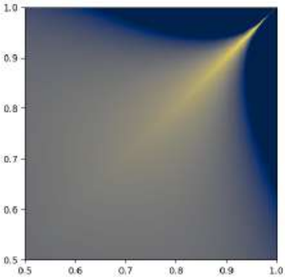
The $\sqrt{n}$ scaling limit of sampled permutations' RS shape when the permuton has a density was also investigated in a recent paper [Sjöstrand '23], thus generalizing Logan-Shepp-Vershik-Kerov's limit curve.

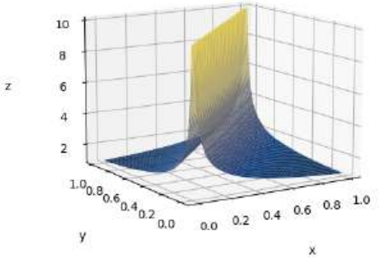
Asymptotics in the case of a diverging density

Theorem [D23+, arXiv: 2301.07658]

For any $\gamma \in (1/2, 1)$, there exist at least two explicit families $\mathcal{F}_\gamma$ and $\mathcal{G}_\gamma$ of densities on $[0,1]^2$, such that for any permuton μ with density $\rho \in \mathcal{F}_\gamma \cup \mathcal{G}_\gamma$, $\mathrm{LIS}(\sigma_n)$ behaves like n^γ up to a logarithmic factor.

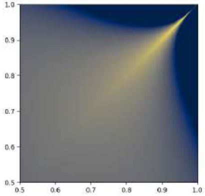
On the left: divergence at a single point.
On the right: divergence along an increasing curve.





Proof ideas

For the first family:

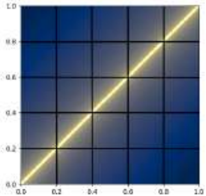


⟷

1. Bound the desired density (on the left) below by a locally constant density (on the right), and above by a mixture of such densities.
2. These simple densities can be studied by applying Vershik-Kerov's result on each box.
3. Then use coupling arguments on the sampled points to deduce bounds on LIS for the desired density.

Proof ideas

For the second family:



1. Slice the unit square into a thin grid, so that the number of points appearing in each box is $O(1)$.
2. Notice that an up-right path of points occupies an up-right path of boxes. Control the number of points appearing in any up-right path of boxes.

Question 1:
If the permuton has a density, $\mathrm{LIS}(\sigma_n)$ is always sublinear. What kinds of permutons yield a linear behaviour of $\mathrm{LIS}(\sigma_n)$?

Question 2:
Recall that the set of permutations is embedded in the space of permutons. Can we extend the functions LIS_k and RS from permutations to permutons?

- Permuton theory and links with permutations
- Asymptotics of LIS and RS for sampled random permutations under density
- The RS shape of permutons, linear asymptotics
- The RS tableaux of permutons, linear asymptotics
- Fomin's inverse algorithm and its continuous analog

The RS shape of permutons, linear asymptotics

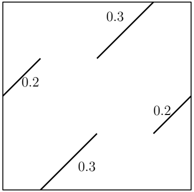
RS shape of a permuton

We say that a subset $A \subset [0,1]^2$ is increasing when for any $(x,y),(x',y') \in A$, one has $(x-x')(y-y') \geq 0$.
If μ is a permuton, define:

$$\widetilde{\mathrm{LIS}}_k(\mu) := \sup_{A_1,\dots,A_k \subset [0,1]^2 \text{ increasing}} \mu(A_1 \cup \cdots \cup A_k)$$

and

$$\widetilde{\mathrm{sh}}(\mu) := \left(\widetilde{\mathrm{LIS}}_k(\mu) - \widetilde{\mathrm{LIS}}_{k-1}(\mu)\right)_{k \in \mathbb{N}^*}.$$



$\widetilde{\mathrm{sh}}(\mu) = (0.6, 0.4, 0, \dots)$

The RS shape of permutons, linear asymptotics

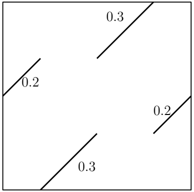
RS shape of a permuton

We say that a subset $A \subset [0,1]^2$ is increasing when for any $(x,y),(x',y') \in A$, one has $(x-x')(y-y') \geq 0$.
If μ is a permuton, define:

$$\widetilde{\mathrm{LIS}}_k(\mu) := \sup_{A_1,\dots,A_k \subset [0,1]^2 \text{ increasing}} \mu(A_1 \cup \cdots \cup A_k)$$

and

$$\widetilde{\mathrm{sh}}(\mu) := \left(\widetilde{\mathrm{LIS}}_k(\mu) - \widetilde{\mathrm{LIS}}_{k-1}(\mu)\right)_{k \in \mathbb{N}^*}.$$



$\widetilde{\mathrm{sh}}(\mu) = (0.6,\, 0.4,\, 0, \dots)$

Rem: if μ has a density then $\widetilde{\mathrm{sh}}(\mu) = (0, \dots)$.

The RS shape of permutons, linear asymptotics

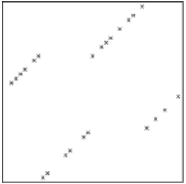
Lemma

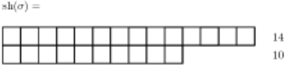
1. It is possible to embed permutations σ into permutons $\mu_\sigma^{\nearrow}$ so that $\widetilde{\mathrm{sh}}(\mu_\sigma^{\nearrow}) = \mathrm{sh}(\sigma)/n$.
2. The function $\widetilde{\mathrm{LIS}}_k$ is *upper semi-continuous*.

Theorem [D23, arXiv: 2307.05768]

If $\sigma_n \sim \mathrm{Sample}_n(\mu)$ then:

$$\frac{1}{n}\mathrm{LIS}_k(\sigma_n) \xrightarrow[n\to\infty]{} \widetilde{\mathrm{LIS}}_k(\mu) \quad \text{a.s.}$$





The RS shape of permutons, linear asymptotics

Theorem [D]

Partial large deviation results:

1. for any $\alpha > \widetilde{\mathrm{LIS}}_k(\mu)$, $\mathbb{P}\big(\mathrm{LIS}_k(\sigma_n) > \alpha n\big) = e^{-n\Lambda^*(\alpha)+o(n)}$;
2. for any $\beta < \widetilde{\mathrm{LIS}}_k(\mu)$, $\mathbb{P}\big(\mathrm{LIS}_k(\sigma_n) < \beta n\big) \simeq e^{-n?+o(n)}$.

To be compared with the uniform case $\frac{1}{\sqrt{n}}\mathrm{LIS}(\sigma_n) \longrightarrow 2$:
upper tail of speed $\sqrt{n}$ and lower tail of speed n.

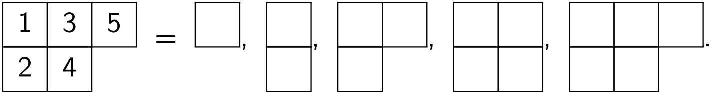
Proof idea for 1:
The most likely scenario for $\mathrm{LIS}_k(\sigma_n) > \alpha n$ to happen is that at least αn points appear in a maximal k-increasing subset of μ
$\rightsquigarrow$ large deviation for sums of i.i.d. Bernoulli random variables.

- Permuton theory and links with permutations
- Asymptotics of LIS and RS for sampled random permutations under density
- The RS shape of permutons, linear asymptotics
- The RS tableaux of permutons, linear asymptotics
- Fomin's inverse algorithm and its continuous analog

The RS tableaux of permutations

Let's change our point of view on the RS tableaux of a permutation.

- A Young tableaux can be seen as a sequence of Young diagrams:



- For any permutation σ and integers i, j, define the word $\sigma^{i,j}$ as the sequence of letters $\sigma(h)$ with $h \leq i$ and $\sigma(h) \leq j$.

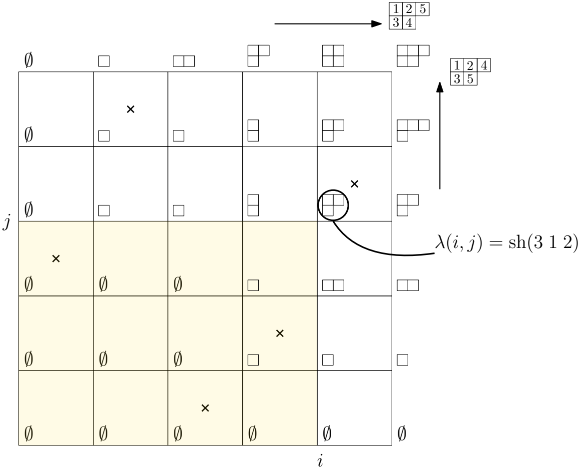
  Ex: $\left(2\ 6\ 3\ 5\ 1\ 4\right)^{4,5} = 2\ 3\ 5$.

  Fact: Let $\sigma \in \mathfrak{S}_n$. For any k, the RS shape of $\sigma^{n,k}$ is the k-th diagram defining $P(\sigma)$ and the RS shape of $\sigma^{k,n}$ is the k-th diagram defining $Q(\sigma)$.

The RS tableaux of permutations

Alternative definition of RS correspondence for permutations

Let $\sigma \in \mathfrak{S}_n$. Then $\mathrm{RS}(\sigma) = \big(\lambda^\sigma(n,\cdot), \lambda^\sigma(\cdot,n)\big)$ where for any integers i,j, $\lambda^\sigma(i,j) = \mathrm{sh}(\sigma^{i,j})$.



The RS tableaux of permutons

Definition

For any permuton μ, define:

$$\widetilde{\mathrm{RS}}(\mu) := \left(\widetilde{\lambda}^{\mu}(1,\cdot), \widetilde{\lambda}^{\mu}(\cdot,1)\right)$$

where for any $(x,y) \in [0,1]^2$:

$$\widetilde{\lambda}^{\mu}(x,y) = \left(\widetilde{\lambda}^{\mu}_{k}(x,y)\right)_{k\in\mathbb{N}^*} := \widetilde{\mathrm{sh}}\left(\mu|_{[0,x]\times[0,y]}\right).$$

The RS tableaux of permutons

Definition

For any permuton μ, define:

$$\widetilde{\mathrm{RS}}(\mu) := \left(\widetilde{\lambda}^{\mu}(1,\cdot), \widetilde{\lambda}^{\mu}(\cdot,1)\right)$$

where for any $(x,y) \in [0,1]^2$:

$$\widetilde{\lambda}^{\mu}(x,y) = \left(\widetilde{\lambda}^{\mu}_k(x,y)\right)_{k\in\mathbb{N}^*} := \widetilde{\mathrm{sh}}\left(\mu|_{[0,x]\times[0,y]}\right).$$

Proposition [D]

$$\frac{1}{n}\lambda^{\sigma_n}(\lfloor nx \rfloor, \lfloor ny \rfloor) \underset{n\to\infty}{\longrightarrow} \widetilde{\lambda}^{\mu}(x,y) \quad \text{a.s.}$$

componentwise, uniformly on $(x,y) \in [0,1]^2$. In particular: convergence of σ_n's RS tableaux after scaling by n.

The RS tableaux of permutons

Done: extending RS to permutons and deducing linear asymptotics for sampled permutations.
Now: nice properties of $\widetilde{\mathrm{RS}}$?

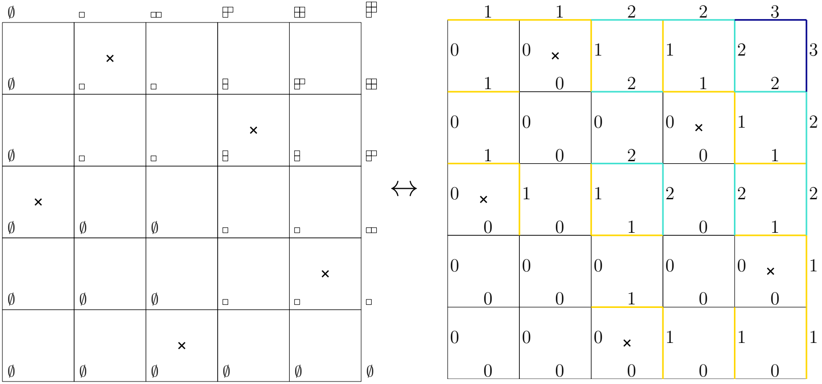
Open question 1:
Is $\widetilde{\mathrm{RS}}$ injective on permutons μ satisfying $\lim_{r\to\infty} \widetilde{\mathrm{LIS}}_r(\mu) = 1$?
Can we invert it?

Open question 2:
Is there an "algorithmic" construction of $\widetilde{\mathrm{RS}}$ or its inverse (it it exists)?

- Permuton theory and links with permutations
- Asymptotics of LIS and RS for sampled random permutations under density
- The RS shape of permutons, linear asymptotics
- The RS tableaux of permutons, linear asymptotics
- Fomin's inverse algorithm and its continuous analog

Fomin's algorithm for permutations

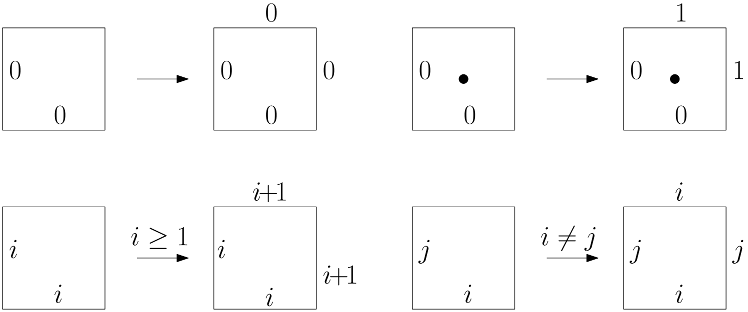
Replace the diagram labels on vertices by integer labels on edges, depending on which row gets a new box:



Fact: $\lambda_k^\sigma(i',j') - \lambda_k^\sigma(i,j)$ is the number of edges labeled k on any up-right path from (i,j) to (i',j').

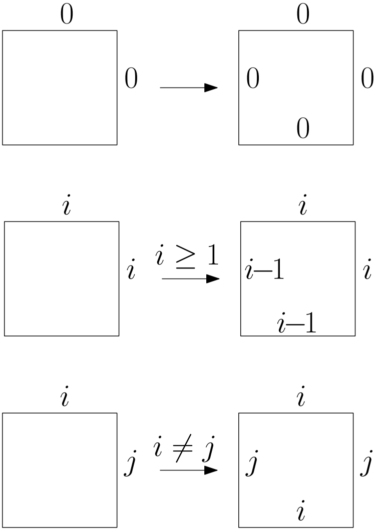
Fomin's algorithm for permutations

The edge labels follow a set of simple local rules (Viennot's version of Fomin's rules):



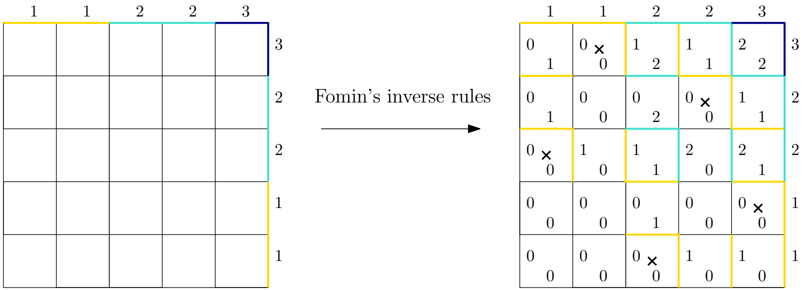
Fomin's algorithm for permutations

The *inverse* local rules are even simpler:



Fomin’s algorithm for permutations

We deduce an algorithm to invert RS correspondence:
Start with the edge labels on north and east borders, successively apply Fomin’s inverse local rules to deduce the whole labeling and the permutation points.



A continuous analog for permutons

Theorem [D23, arXiv: 2307.05768]

Suppose $\widetilde{\mathrm{LIS}}_r(\mu) = 1$ for some $r \geq 1$. Let $(x, y) \in (0,1]^2$ such that for any $1 \leq k \leq r$, the following left-derivatives

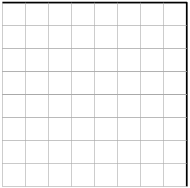
$$\alpha_k := \partial_x^- \widetilde{\lambda}_k^\mu(x,y) \quad \text{and} \quad \beta_k := \partial_y^- \widetilde{\lambda}_k^\mu(x,y)$$

exist. Then for any $s, t \geq 0$ and $1 \leq k \leq r$:

$$\lim_{\epsilon \to 0^+} \frac{\widetilde{\lambda}_k^\mu(x,y) - \widetilde{\lambda}_k^\mu(x - t\epsilon, y - s\epsilon)}{\epsilon} = \phi\big((t\alpha_i)_{k \leq i \leq r}, (s\beta_i)_{k \leq i \leq r}\big)$$

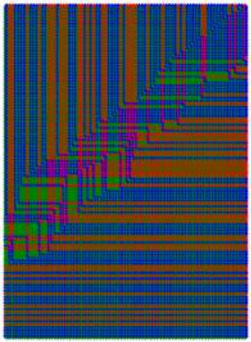
where ϕ is a certain continuous function.

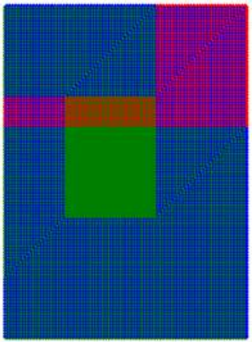
Locally and asymptotically for $\sigma_n \sim \mathrm{Sample}_n(\mu)$:



A continuous analog for permutons: proof ideas

On the left, Fomin's inverse rules applied to randomly distributed edge labels (0, 1, 2) on north and east borders.
On the right, north and east borders have been ordered with higher labels closer to north-east corner.

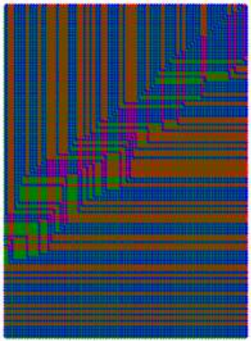




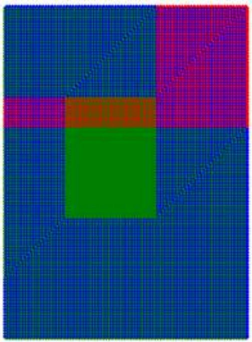
A continuous analog for permutons: proof ideas

On the left, Fomin's inverse rules applied to randomly distributed edge labels (0, 1, 2) on north and east borders.
On the right, north and east borders have been ordered with higher labels closer to north-east corner.

21 1's and 10 2's $\rightarrow$



20 1's and 10 2's $\rightarrow$



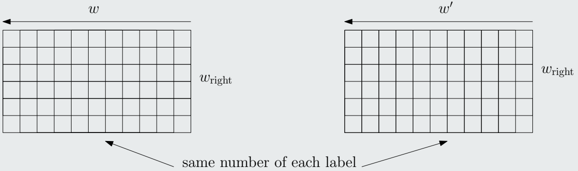
Different dynamics, but $\simeq$ same number of each edge label on south and west borders!

A continuous analog for permutons: proof ideas

Two equivalence relations on words

- w, w' are *Fomin equivalent* when they always yield the same number of each edge label after Fomin's inverse algorithm.

$\forall w_{\text{right}},$



- w, w' are *Knuth equivalent* when they differ by a sequence of elementary transformations:

$$\begin{cases} j\,i\,k \longleftrightarrow j\,k\,i & \text{if } i < j \leq k; \\ i\,k\,j \longleftrightarrow k\,i\,j & \text{if } i \leq j < k. \end{cases}$$

A continuous analog for permutons: proof ideas

Theorem [D]

Knuth equivalence implies Fomin equivalence.

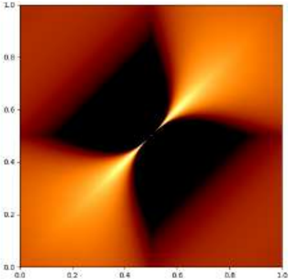
Theorem

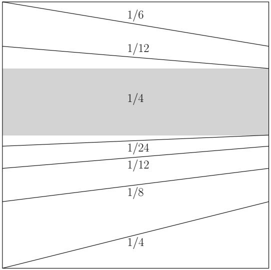
Two words are Knuth equivalent iff they have the same P-tableau.

It is easier to prove that the P-tableau of a random word (bounded letters, diverging size) is similar to the P-tableau of its decreasing reordering!

+ other technical properties...

Application: invert μ's RS tableaux?





Thank you for your attention!